Trading Favors

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1 Introduction

There are many real life situations in which large groups of agents cooperate even though any particular pair of agents interacts only rarely with one another. For example, Granovetter (1974) analyzed how workers get matched to new jobs and showed that a majority of workers found new jobs through referrals provided by 'weak links' with other agents (see Granovetter (1973)). Such a referral frequently involves agents with whom the recipient of the referral interacts only at a very low frequency.

If interactions are analyzed in a Prisoner's Dilemma game played between the sender and the recipient of a favor, the obvious question is how can cooperation be sustained when agents interact only infrequently. The existing literature has provided two possible answers to this question.

For any population size, cooperation can be generally enforced if group punishments are available. In many settings, institutions have developed to help groups of agents aggregate information efficiently. An example are diamond traders who put up pictures of non-trustworthy individuals in trading rooms.[1] Traders can then identify defectors easily and punish them by refusing to do further business with them. A related literature on indirect reciprocity has emphasized the importance of 'image scores' in enforcing indirect reciprocity (see, for example, Nowak and Sigmund (1998)).

If group punishments are not available and agents only have information about the outcomes of past personal encounters with other agents then cooperation can

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[1] I thank Paul Milgrom for this example.

generally not be enforced for large population and fixed discount factors as Kandori (1992) has shown.

In this paper I present a model in which cooperation remains sustainable even as the frequency with which agents interact tends to zero and in the absence of institutions which aggregate information about the conduct of all agents. In my model agents develop needs for 'favors' at a fixed rate. Each agent in the society can provide any favor only with a small probability p and the ability to provide a favor is private information. This implies that any pair of agents can provide bilateral help only at very low frequency. As $p \to 0$ bilateral support becomes unsustainable as we would expect in the light of Kandori (1992). I then add the possibility for agents to provide *indirect* favors. This implies that an agent who cannot provide a favor herself can send a request to some acquaintance who 'owes' her a favor. This acquaintance can then in turn relay the favor to a third agent who owes him a favor and so on. When such a chain of favors and relayed favors 'clears' each intermediate agent basically trades one favor for another - therefore he does not lose by relaying favors.

Indirect favors effectively increase the frequency with which agents can help each other, and thus assist in stabilizing bilateral cooperation between agents. Alternatively, we can think of providing a favor to another as opening a 'link' to this agent who then owes a favor to the original provider. Indirect favors make such directed links more valuable because they give access not only to the recipient of the actual favor, but also to all his indirect links which potentially connect the agent to a very large number of other agents. This interpretation of the value of a link is reminiscent of Jackson and Wolinsky (1996) who analyzed the stability of links in locally connected graphs in a cooperative game theory setting.

The paper is organized as follows. Section 2 analyzes a simple model with two agents. If the ability to provide a favor is unobservable agents naturally make the provision of a favor conditional on whether they 'owe' a favor (i.e. they provided fewer favors in the past than they received). Section 3 extends the two agent model to a multi-agent setting and allows indirect favor provision. The indirect favor model is then analyzed in section 4. In section 5 results from simulations are presented.

2 Equilibrium in a Model with Direct Favors

I start the analysis with a simple two agent model. Time is continuous and agents discount future utility at rate r. Each agent develops a need for a favor $\tilde{f} = f_{i,t}$ at rate 1. No favor can be provided by the agent herself. A granted favor has utility b to the receiver and imposes a cost $c < b$ to the sender. I assume that an agent can provide a needed favor with probability $0 < p \leq 1$.

If the ability to provide favors is observable by both agents a simple equilibrium of the model looks as follows:

1. Agents grant favors whenever possible.

2. If an agent sees the other agent defect (i.e. not help when he can) he does not grant further favors in the future.

These strategies form an equilibrium for

$$\frac{p\,(b-c)}{r} > c. \tag{1}$$

However, I impose the following assumption instead.

Assumption 1 *The ability to provide a favor is private information.*

Agents now have to infer from the history of favor exchange to what extent their partner provided favor for them. A particularly simple and intuitive accounting device is the difference k_i in the past number of favors provided by agent i to agent j versus the number of favors she received from agent j. Clearly, we have $k_i = -k_j$.

This state variable is an attractive choice to anchor agent's strategies because it captures the common notion that someone 'owes' a favor (negative k) or is owed a favor (positive k). In this section I focus on equilibria where agents' strategies only depend on the state variable k_i.[2]

2.1 Symmetric Equilibria and Value Function

I denote the value of an agent relationship with his acquaintance in state k with V_k and consider symmetric equilibria where $-K \leq k \leq K$ with the following strategies:

1. Agents grant favors if $k < K$.

2. Agents stop granting favors if $k = K$.

The Bellman equation and boundary conditions can be written as:

$$\begin{aligned} rV_k &= p\,(V_{k+1} - V_k - c) + p\,(V_{k-1} - V_k + b) \\ rV_K &= p\,(V_{K-1} - V_K + b) \\ rV_{-K} &= p\,(V_{-K+1} - V_{-K} - c) \end{aligned} \tag{2}$$

[2]These equilibria will generally not be optimal if agents also have access to information about the order and the time with which they received passed favors. The optimal equilibrium looks much more complex because the state space becomes continuous and multi-dimensional.

This equation system defines a Markov perfect equilibrium if the incentive compatibility constraints and the individual rationality constraints are fulfilled. First, agents have to gain from doing favors for $k < K$, and second, the relationship between two agents must have positive value for $k = -K$:

$$\begin{array}{ll} \text{(IC)} & V_{k+1} - V_k > c \quad \text{for } -K \leq k < K \\ \text{(IR)} & V_{-K} > 0 \end{array} \tag{3}$$

I introduce two helpful parameters to present the next theorem:

$$x_1 = 1 + \frac{r}{2p} - \sqrt{\left(\frac{r}{2p}\right)^2 + \frac{r}{p}} \tag{4}$$

$$z = x_1^{2K+1} \tag{5}$$

Theorem 1 *A Markov perfect equilibrium of the two agent model exists if and only if*

$$z > z^* = \frac{b}{c} - \sqrt{\left(\frac{b}{c}\right)^2 - 1} \tag{6}$$

Proof: see appendix A

The equilibrium is clearly second-best from the social planner's point of view because both agents only grant favors if the state k is bounded away from the two boundaries, i.e. $-K < k < K$. By increasing K agents' utility from the relationship increases because they spend less time at the boundary of the state space. Equation 6 allows us to calculate the optimal symmetric equilibrium with maximal $\overline{K}\left(\frac{r}{p}, \frac{b}{c}\right)$:

$$x_1^{2\overline{K}\left(\frac{r}{p}, \frac{b}{c}\right)+1} = z^* \tag{7}$$

We know that $\overline{K}$ is an increasing function in x_1 and decreasing in z^*. We also know that $\frac{dx_1}{d\frac{r}{p}} < 0$ and $\frac{dz^*}{d\frac{b}{c}} < 0$.

Corollary 1 *The maximal support* $\overline{K}\left(\frac{r}{p}, \frac{b}{c}\right)$ *satisfies* $\frac{\partial \overline{K}}{\partial \frac{r}{p}} < 0$ *and* $\frac{\partial \overline{K}}{\partial \frac{b}{c}} > 0$

Figure 1 shows some plots of $\overline{K}(p)$ for various parameters. It is intuitive that K increases as p increases: a higher arrival rate of favors makes agents more willing to help because cheating becomes harder. Figure 2 plots the value function for various cases. Note, that all value functions in the figure are concave which can be shown generally for all solution of the Bellman equation 2 and which satisfy incentive compatibility.

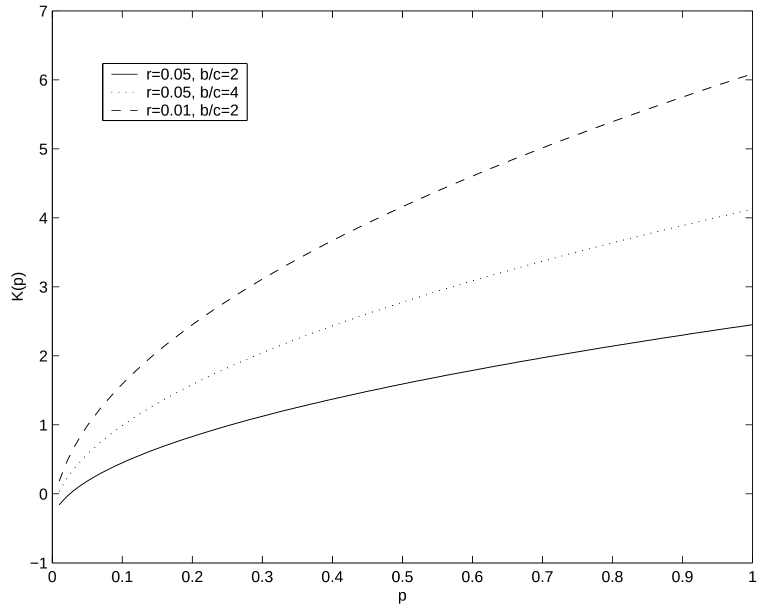


Figure 1: Maximal support $\overline{K}(p)$ for various combination of $(r, b/c)$

The final result tells us that favor exchange is easier to sustain for larger benefit-cost ratio of favors but that it always harder to cooperate if agents do not know if their partner can provide a favor. Note, that under full information always providing favors is an equilibrium for $\frac{b}{c} > \frac{r}{p} - 1 = D_{FI}\left(\frac{r}{p}\right)$.

Corollary 2 *Under assumption 1 (limited) favor exchange can be supported for*

$$\frac{b}{c} > D\left(\frac{r}{p}\right) > D_{FI}\left(\frac{r}{p}\right). \tag{8}$$

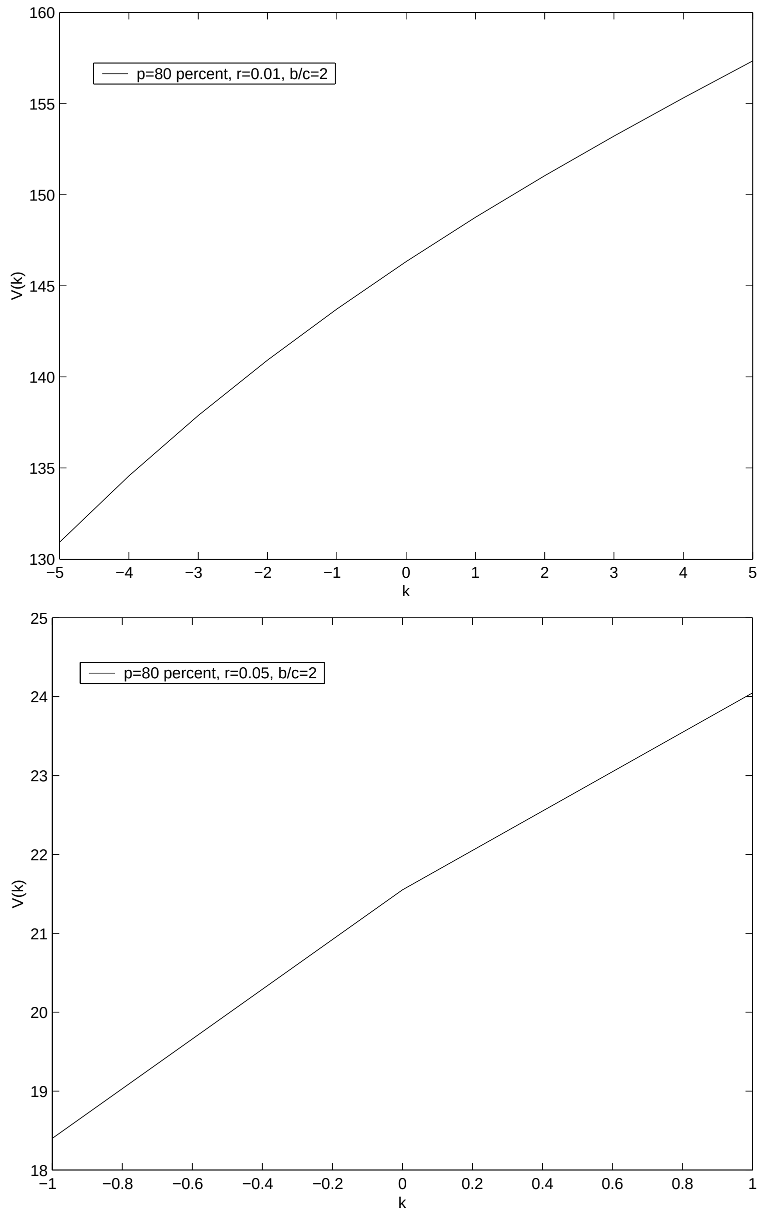
The function $D\left(\frac{r}{p}\right)$ *is increasing in* $\frac{r}{p}$ *and* $D\left(\frac{r}{p}\right) \to \infty$ *as* $\frac{r}{p} \to \infty$.

Remark 1 *Note that cooperation becomes unsustainable both under assumption 1 and under perfect information as* $p \to 0$. *This is intuitive: a partner is only useful if he can at least sometimes provide favors. In contrast, in our model with indirect favors cooperation can be supported even as* $p \to 0$.

2.2 Simplified Framework with Binary State Space

Equilibrium strategies which rely on the difference between the number of favors provided and the number of favors received are intuitive. However, when I study

Figure 2: Optimal (second-best) value functions for $r = 0.01$ and $p = .8$ (top), and $r = 0.05$ and $p = .8$ (bottom)



indirect favors in the next section even this state space becomes cumbersome to work with. I therefore consider equilibria with an even simpler state space: agents condition their action simply on whether they provided the last favor or their partner did so. I define a binary state variable $B(i,j)$ as follows:

$$\begin{array}{ll} B(i,j) = 1 & \text{if agent } i \text{ provided the last favor} \\ B(i,j) = 0 & \text{if agent } j \text{ provided the last favor} \end{array} \tag{9}$$

Note that $B(i,j) + B(j,i) = 1$.

The Bellman equations for the problem are:

$$rV_1 = p\left(V_0 - V_1 + b\right) \tag{10}$$

$$rV_0 = p\left(V_1 - V_0 - c\right) \tag{11}$$

It can be easily checked that:

$$V_1 - V_0 = \frac{p(b-c)}{r+2p} \tag{12}$$

$$V_0 = \frac{p}{r}\left[\frac{p(b-c)}{r+2p} - c\right] \tag{13}$$

The incentive compatibility constraint $V_1 - V_0 > c$ and the individual rationality constraint $V_0 > 0$ collapse to the single condition:

$$\frac{p(b-c)}{r+2p} > c \tag{14}$$

This implies that exchanging favors can be an equilibrium even if agents only remember whether they 'owe' a favor or not.

Lemma 1 *Under assumption 1 and a binary state space (limited) favor exchange can be supported for*

$$\frac{b}{c} > D_B\left(\frac{r}{p}\right) > D_{FI}\left(\frac{r}{p}\right). \tag{15}$$

3 A Simple Model of Indirect Favors

I now extend the model to a multi-agent setting with indirect favors. as in Kandori (1992). I assume that agents cannot observe the exchange of favors between other agents, i.e. there are no institutions such as 'image scores' which aggregate information and make group punishments possible.

Assumption 2 *Agents cannot observe the exchange of favors between other agents.*

I also make a simplifying assumption on the information available to agents about the past history of received and provided favors.

Assumption 3 *Any two agents can only recall the last favor which was exchanged between them.*

The information two agents share about each other is therefore essentially binary. This makes bilateral favor exchange with a binary state space still possible. However, the richer equilibria where agents condition their strategies on the difference between provided and received favors are no longer sustainable. Note that because of assumption 2 agents do not know the state of their acquaintances, and their indirect acquaintances. Therefore, they have to predict the state of their neighbors. Any information about the frequency and the order of past favor exchanges will influence this prediction. Hence, the state space of each agent would be very complicated. Assumption 3 allows us to summarize the state of an agent by the number of her open links.

I propose an equilibrium where each agent interacts with a small number of n 'acquaintances' with whom she exchanges both direct and indirect favors. I do not describe how agents meet acquaintances in the first place. Such a story can be provided easily, and is complementary to the model that I develop in this section. For example, agents could randomly match in some initial 'link formation' stage under the convention that two agents only become acquaintances when they mutually exchange 'gifts' (as in Carmichael and MacLeod (1997)). By determining the right size of the gift one can ensure that agents would only acquire a limited number of acquaintances because the marginal value of each acquaintances decreases with the total number of acquaintances.

3.1 Demand and Production of Favors

There is a continuum of agents indexed by $i \in [0,1]$. As in the two-agent model time is continuous and agents discount the future at rate r. At rate 1 each agent demands a favor $f_{i,t}$. Receiving the favor gives the agent utility b and imposes a disutility $c < b$ on the sender. Providing favors is therefore socially optimal. We assume that the need for the favor disappears at rate ρ and we will concentrate on equilibria as $\rho \to \infty$.[3] Therefore, favors have to be provided as quickly as the need arises in order to be useful.

Any agent in the economy can independently provide a particular favor $f_{i,t}$ with probability $0 < p < 1$. In this section I will concentrate on the interesting limit case $p \to 0$. Bilateral favor exchange is then never privately optimal, and there will be no cooperation in equilibrium even though the social planner could

[3] This is a technical assumption and will assure that agents have no incentive to send a request for a favor more than once over their network.

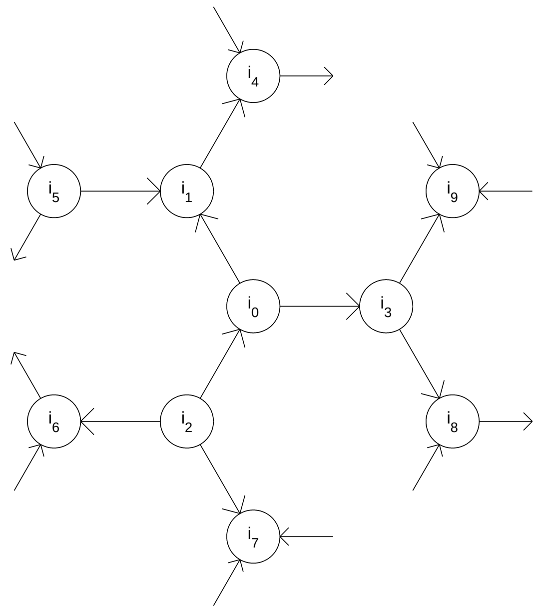


Figure 3: Graph with acquaintance number $n = 3$

continue to match up each need for a favor with a potential sender since a share $p > 0$ of all agents are able to provide some favor $f_{i,t}$.

3.2 Social Network

Each agent has n acquaintances who are randomly selected from the uniform distribution on $[0, 1]$. The acquaintance relationship is symmetric, i.e. if i is an acquaintance of j then j is also an acquaintance of i. The acquaintance relationship defines a random graph $G \subset [0, 1] \times [0, 1]$ where each link $l = (i, j) \in G$ represents the relationship between two acquaintances.

Since there is a continuum of agents the random graph G has no 'loops' , i.e. a finite chain of indirect links starting from some agent i never includes i herself.[4]

I define a path as a finite sequence (tuple) of agents $\theta = (i_0, i_1, .., i_k)$ such that successive agents in the sequence are acquaintances, i.e. $(\phi(m), \phi(m + 1)) \in G$. Paths allow me to define a natural geometry on the graph: the R-neighborhood $N_{i,R}$ of agent i is the set of all agents which agent i can reach through a path of length at most R. It is easy to see that $|N_{i,R}| = n \times n^{R-1}$.

[4]More precisely, such an event has zero probability.

3.3 Open and Closed Bonds

In equilibrium, each link $l = (i, j)$ can be 'open' (state 0) or 'closed' (state 1). Formally, a function $B : G \to \{0, 1\}$ describes the state of each link and makes the graph G directed. If agent i has an open link to agent j, then agent j's link to i is closed, i.e.

$$B(i, j) + B(j, i) = 1. \tag{16}$$

I interpret an open link between agent i and j as agent i 'owing' a favor to agent j. Graphically an open link between agent i and j is represented by an arrow which points away from agent i towards agent j. In figure 3, for example, agent i_0 owes a favor to agent i_2 and is owed favors by agents i_1 and i_3.

A path ϕ is said to be an open path if it is a path and each successive link is open ,i.e. $B(\phi(m), \phi(m+1)) = 1$. The set of open paths originating from some agent i is denoted with $\Phi(i)$. The number of open paths will be crucial to make cooperation sustainable in equilibrium.

Although each agent is connected to only n acquaintances he can be indirectly connected through open paths to infinitely many agents. To see this, consider a graph where for every agent i each of her n links are open and closed independently with probability $\frac{1}{2}$. The probability that a particular open link (i, j) connects agent i to infinitely many agents is q. Neighbor j has $0 \leq k \leq n-1$ open links with probability $\binom{n-1}{k} 2^{1-n}$. The following recursive equation can be used to calculate q:

$$q = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{1-n} \left[1 - (1-q)^k\right] = F_n(q) \tag{17}$$

Then $F(0) = 0$, $F(1) < 1$ and

$$F'(0) = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{1-n} k \tag{18}$$

which is greater than 1 for $n > 3$. Therefore, agents in a random bond network have indirect access to infinitely many agents with probability $q > 0$ for $n > 3$.

3.4 Direct and Indirect Favors

Agents can *costlessly* send messages $m_S = \tilde{f}$ to their acquaintances to ask for a favor $\tilde{f}$.[5] For every message the receiver i_1 only knows which neighbor i_0 asked for a favor $\tilde{f}$, i.e. they know the pair $m_R = \left(i_0, \tilde{f}\right)$.

When observing a request for a favor agents can take three possible actions:

[5] The results would not change if agents could also send messages to non-acquaintances. In equilibrium such messages would simply be discarded.

1. They can ignore the favor.

2. If they can provide the favor (with probability p) they can send the favor $\tilde{f}$ to the sender i after some (short) waiting time $\frac{\epsilon}{\rho}$.

3. They can 'relay' the favor by sending it through an open link to some other acquaintance i_2. For simplicity, I assume that recipients cannot distinguish between direct and indirect favors. However, this convention does not matter for the results. Indirect favors travel along the set of open paths $\Phi(i)$ (which is potentially infinite as we have seen above), and form a sequence of chains (i_0, i_1),(i_0, i_1, i_2),..,$(i_0, i_1, .., i_k)$. Whenever some agent i_k along such a chain provides the favor $\tilde{f}$ it 'clears' immediately along the chain: agent i_k provides the favor to agent i_{k-1} who in turn provides it i_{k-2} until it reaches the original sender i_0.[6] If two or more agents can provide a specific favor $\tilde{f}$ at the same time a tie-breaking rule applies in which each agent provides the favor with equal probability.

A positive reaction time $\frac{\epsilon}{\rho} > 0$ is necessary because agents need time to respond to messages. However, I will usually analyze equilibrium behavior under the assumption that $\epsilon \to 0$. This will guarantee that favors are granted 'almost' immediately even if they have to be relayed many times.[7]

3.5 'Swapping' Acquaintances

At rate a pairs of linked acquaintances $l = (i_1, i_2)$ and $l' = (i_3, i_4)$ are randomly matched. All agents involved derive utility $\frac{B}{a}$ from 'recombining' such that $B > b$. Agents can recombine in two ways:

$$K_1 = \{(i_1, i_3), (i_2, i_4)\} \quad (19)$$
$$K_2 = \{(i_1, i_4), (i_2, i_4)\} \quad (20)$$

I assume that agents are indifferent between both recombinations. Note, that only one recombination preserves the orientation of bonds for all agents. A coordination game with the following structure determines how agents recombine and how the bond orientation is affected:

1. Each agent makes an announcement $\tilde{b}_i \in \{0, 1\}$ about her bond orientation. Announcement $\tilde{i} = 1$ indicates that she has an open bond with her acquaintance.

[6] A more realistic specification would allow favors some time to travel back along the chain.

[7] If an agent can reach infinitely many agents through open paths the expected waiting time for providing a favor is $\frac{\epsilon}{\rho p}$ which tends to zero as $\epsilon \to 0$. Note that the waiting time for a favor tends to zero faster than the rate at which the need for the favor disappears.

2. Agent i observes announcements $\tilde{b}_{-i}$ and simultaneously proposes to match with agent $k_i\left(\tilde{b}_{-i}\right)$ and adopt bond orientation $b_i\left(\tilde{b}_{-i}\right)$.

3. Agents recombine if and only if their announcements are compatible: agents mutually agree on whom to match with and on the subsequent bond orientation. For, example if agents agree to rematch according to K_1 the rematch will be performed if and only if $b_{i_1} + b_{i_3} = 1$ and $b_{i_2} + b_{i_4} = 1$.

4 Equilibrium in the Indirect Favor Model

In this section I analyze an equilibrium of the model with indirect favor provision as $p \to 0$. It should be emphasized that bilateral, direct favor provision is never privately optimal for small values of p. However, indirect favor provision makes individual open links more valuable because agents can potentially reach infinitely many agents through open paths. I start by defining equilibrium strategies of all agents, and then prove that it satisfies a set of equilibrium conditions.

The equilibrium strategies of agents are simple and intuitive.

Definition 1 *In the indirect favor equilibrium (a) agents provide favors whenever possible, and (b) both send and relay all favors which they cannot provide through as many open links as possible according to the following rules:*

1. *An agent i who needs a favor $\tilde{f} = f_{i,t}$ immediately sends requests for help through all his open links.*

2. *An agent j who receives a message $\left(i, \tilde{f}\right)$ from agent i and can provide a favor will do so.*

3. *An agent j who receives a message $\left(i, \tilde{f}\right)$ from agent i and cannot provide a favor will resend it through each of his open links (if he has any).*

4. *If agent i provides a favor to agent j then agent i the orientation of the link between i and j becomes 'open' (i.e. i 'owes' a favor to j).*

5. *Agents announce their type truthfully in the recombination game, and propose compatible matches: if agent i_1 has an existing 'open' link to his old partner i_2 and can rematch with either i_3 and i_4 he will choose the agent with the 'closed' link as his preferred partner.*

Before the equilibrium conditions are discussed I would like to emphasize two points. First of all, each agent i's knowledge about the state of system is summarized in the number of open links $0 \leq m_i \leq n$ due to assumption 3. I will also refer to m_i as the state variable of agent i. This simplifies the analysis enormously because one just has to check that the equilibrium strategies are optimal for any agent in one of those $n+1$ possible states. Second, indirect favor provision creates trade in favors. When a favor is finally provided by some agent i_k and clears along some open path $(i_0, i_1, .., i_k)$, each intermediate agent i_h $(0 < h < k)$ essentially trades his open favor to agent i_{h+1} for an open favor to agent i_{h-1}. Therefore, the state m_{i_h} of any intermediate agent does not change in the process.

4.1 Swapping Acquaintances

Lemma 2 *Swapping friends is always optimal for agents.*

Proof: see appendix

Swapping friends is a dominant strategy for agents because the flow benefit B from swapping friends exceeds the maximal benefit which can be derived from a link. However, swapping friends does not affect the state m_i of any agent i.

4.2 Optimality of Favor Trading

Although trade in favors does not affect the states of intermediate agents along an open path, it is not obvious whether an agent should trade one of his open links for a closed link. His willingness to do so will depend on the probability $q(m_i)$ of receiving a favor in the future when sending it through the existing open link, and the probability $q^*(m_i)$ of receiving a favor in the future when sending it through the (currently) closed link which will become an open link after the trade in favors has been accomplished.

Lemma 3 *In equilibrium,* $q^*(m_i) > q(m_i)$.

Proof: see appendix

The intuition for this result is as follows. Assume agent i has an open link to agent j and a closed link to agent j'. This implies that agent j owes *at least* one favor while agent j' is owed at least one favor. Agents have needs arising at rate 1. Therefore, it is more likely that agent j had to use up his open links in order to obtain favors than agent j'. Therefore, by trading an open link to j for an open link to j' agent i gains on average access to *more* open paths through agent j's open links. Therefore, agent i gains on average from trading his favor.

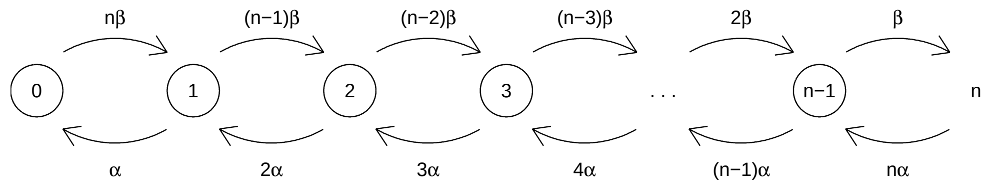


Figure 4: Markov transition matrix when open links give access to finitely many agents

4.3 Access Provided by Open Links

I next calculate the access $\hat{q}(m_i)$ provided by an open link which I defined as the probability that a favor sent through it will be granted through some open path. The probability $\tilde{q}(m_i)$ that an agent can reach infinitely many other agents through an open link as $p \to 0$ is of main interest. There are two cases to distinguish: $\underline{\tilde{q}} = \min_{m_i} [\tilde{q}(m_i)] = 0$ and $\underline{\tilde{q}} > 0$. It is not difficult to exclude the former case.

Proposition 1 *For $a > \underline{a}$ and $n > 3$ there is a positive probability that an agent can reach infinitely many agents through some open link as $p \to 0$.*

Proof: see appendix

To understand the intuition behind the proof it should first be noted that the state of the agent as $a \to \infty$ can be suppressed when calculating $\tilde{q}(m_i)$. Next, I assume that agents reach infinitely many other agents through an open link with probability $\tilde{q} = 0$ and show that this leads to a contradiction.

Assume that the equilibrium share of agents in state m is x_m. Agent i who has an open link to agent j cares about how many open links $0 \leq \tilde{m} < n$ his acquaintance has. Let us denote agent i's estimate of agent j having $\tilde{m}$ open links with $y_{\tilde{m}}$. As $a \to \infty$, $y_{\tilde{m}}$ can be (approximately) calculated as follows:

$$y_{\tilde{m}} = \frac{x_{\tilde{m}}}{1 - x_n} \tag{21}$$

I can then compute the expected number α of agents who can be reached through an open link. For small p and finite α the probability of obtaining favors is approximately proportional to the number of open links. Similarly, opportunities to do favors arise at rate β through every closed link and the opportunity to do a favor is therefore proportional to the number of closed links. This allows us to define a transition matrix between the $n+1$ possible states for each agent as shown in figure 4. In equilibrium the expected number of open links has to be $\frac{n}{2}$ because each open link is a closed link for the acquaintance. It can be easily checked that

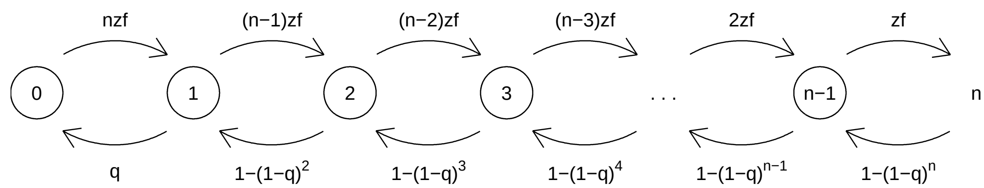


Figure 5: Markov transition matrix when open links give access to infinitely many agents

this implies that $\alpha = \beta$ and $x_m = \binom{n}{m} 2^{-n}$, i.e. each link is open and closed independently. As has been shown in section 3.3, this implies that each agent has access to infinitely many agents with positive probability which is a contradiction.

The next proposition characterizes the equilibrium of the favor giving model.

Proposition 2 *For $a > \underline{a}$ and $n > 3$ a favor which is sent through an open link is provided with positive probability $\tilde{q}(m_i) = q + O(a^{-1}) > 0$ as $p \to 0$. Each closed link receives an opportunity to do a favor at rate $\tilde{z}(m_i) f = zf + O(a^{-1})$. The share x_m of agents in state m is determined by a Markov transition matrix in figure 5. The parameters z, f and q are determined through the following set of equations:*

$$(1 - x_n) q = \sum_{m=0}^{n-1} x_m [1 - (1 - q)^m] \tag{22}$$

$$(1 - x_0) z = \sum_{m=1}^{n} x_m \left[1 - (1 - z)^{n-m}\right] \tag{23}$$

$$\sum_{m=0}^{n} m x_m = \frac{n}{2} \tag{24}$$

Proof: see appendix

The parameter q denotes the probability that an open link gives access to infinitely many favors and is calculated in equation 22 (see section 3.3 for details). Similarly, the parameter z denotes the probability that infinitely many agents have access to an agent's closed links and is calculated in equation 23. Equation 24 is the familiar adding up constraint for open links since exactly half of all bonds have to be open.

The transition matrix in figure 5 can now be explained. Each agent i in some state m_i develops needs for favors at rate 1. Effectively, she will only be able to satisfy these needs when at least one of her links gives her access to infinitely many agents (in this case she will receive a favor for sure). In state m_i she will have access to infinitely many agents with probability $1-(1-q)^m$. The flow of successfully granted favors in equilibrium is:

$$F = \sum_{m=0}^{n} x_m \left[1-(1-q)^m\right] \tag{25}$$

An agent will be able to provide favors through a closed link if she is connected through 'closed' paths to infinitely many agents. Conditional on having access she will have the opportunity to do favors at some rate f. Hence, each closed link 'switches' to being open independently at rate zf. This finishes the description of the Markov transition matrix in figure 5. Note, that in equilibrium favors in the entire population are granted at rate

$$G = \frac{n}{2} zf \tag{26}$$

Using condition 24, I finally check that the model is internally consistent, i.e. the rate F at which favors are received equals the rate G at which favors are given:

$$F = \sum_{m=1}^{n} x_m \left[1-(1-q)^m\right] = \sum_{m=0}^{n} x_m \left(n-m\right) zf = \frac{n}{2} zf = G \tag{27}$$

Remark 2 *At this point the role of swapping acquaintances at some rate a should be clear. Essentially, recombinations remove the correlation between the states of neighboring agents, and allow me to calculate the parameters q and z in equation 22 and 23. Numerical simulations show that neither the qualitative results nor the quantitative results are affected by removing the ability to recombine (i.e. set* $a=0$*). It should therefore be possible to prove all results in this paper without this rather unattractive assumption.*

4.4 Optimality of Favor Giving

I have shown that agents can still receive and give favors at high frequency even as the frequency with which agents can provide bilateral favors tends to zero. I now provide the final result.

Theorem 2 *For* $a > \underline{a}$*,* $n > 3$ *and* $\frac{b}{c} > D(r)$ *(*$D'(r) > 0$*) the strategies in definition 1 form a Markov perfect Nash equilibrium even as* $p \to 0$*.*

Proof: see appendix

As in section 2 I derive a value function V_m which has to fulfill the Bellman equations:

$$\begin{aligned} rV_m &= (n-m)\,zf\,(V_{m+1}-V_m-c)+ \\ &+ [1-(1-q)^m]\,(V_{m-1}-V_m+b) \quad \text{for } 0<m<n \end{aligned} \tag{28}$$

$$rV_n = [1-(1-q)^n]\,(V_{n-1}-V_n+b) \tag{29}$$

$$rV_0 = nzf\,(V_1-V_0-c) \tag{30}$$

As before that the resulting value function can be shown to satisfy both the IC constraint and the IR constraint for sufficiently large benefit-cost ratio $\frac{b}{c}$:

$$\begin{array}{ll} \text{(IC)} & V_{m+1}-V_m>c \quad \text{for } 0\leq m<n \\ \text{(IR)} & V_0>0 \end{array} \tag{31}$$

Remark 3 *Note that having more acquaintances does not necessarily result in more cooperation. On the one hand, more acquaintances increase the probability that agents have access to infinitely many open paths. On the other hand, more acquaintances reduce the incentives to actually provide a favor when asked because the IC and IR conditions will be harder to fulfill.*

5 Numerical Simulations

The assumption that agents 'swap' acquaintances at rate $a>0$ was included as a technical necessity and does not fit well into the rest of the model. Ideally, I would like to extend the results to the case $a=0$ and leave the assumption out entirely. This section presents several numerical simulations that illustrate the results of the previous section, and also demonstrate that variations in the parameter value a do not affect the qualitative results at all.

For small a one cannot neglect the effect of the state m_i on the probability $q(m_i)$ that an agent i can send a favor through an existing open link. The reason is that agent i's state m_i is positively correlated with his neighbor's state. This can be seen most easily for small m_i. In this case agent i will channel a lot of requests for help through his open link to his acquaintance j. When agent j has many open links himself he is more likely to grant a direct or indirect favor while agents with fewer open links are less likely to do so. This implies that agent i gets 'stuck' for longer time periods with agents with few links. In contrast, when agent i has only open links he will never relay any indirect requests for favors.

This positive correlation effect is present in simulations as table 1 shows. The table breaks down the share of agents in state m_i for the case $a=40$ and $a=0$ separately. Furthermore, the table lists the expected number of open links by

Table 1: Comparison of state distribution and favor giving for different rates a of acquaintance 'swapping'. (10,000 agents, $n = 4$, $p = 0.005$)

Swapping rate $a = 40$

	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
x_m	0.098	0.244	0.313	0.251	0.094
$Q^{open}(m)$	NA	1.34	1.37	1.38	1.37
$Q^{closed}(m)$	2.62	2.62	2.64	2.65	NA

Swapping rate $a = 0$

	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
x_m	0.080	0.245	0.349	0.246	0.079
$Q^{open}(m)$	NA	1.32	1.41	1.47	1.50
$Q^{closed}(m)$	2.49	2.54	2.59	2.67	NA

The share of agents in state m is denoted with x_m. If agent i in state m_i has an open (closed) link to agent j then this acquaintance will have $Q^{open}(m)$ ($Q^{closed}(m)$) open links himself on average.

acquaintances of agent i in state m_i depending on whether the link to the acquaintance is open or closed ($Q^{open}(m)$ and $Q^{closed}(m)$). The simulations also reveal that 'trading favors' is indeed a winning strategy as $Q^{open}(m) < Q^{closed}(m) - 1$.

The table demonstrates that the positive correlation in the states of acquaintances does not affect the qualitative results. Both in the presence of 'acquaintance swapping' ($a = 40$) and in the absence of recombination the share of favors which get fulfilled is at about 72 percent on average. These observation suggest that the results in this paper should remain valid if we drop the recombination assumption and set $a = 0$.

6 Conclusion

References

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A Proof of Theorem 1

$$V_k = \frac{p\left(b-c\right)}{r} + Cx_1^k + Dx_2^k \tag{32}$$

where $x_{1/2} = 1 + \frac{r}{2p} \pm \sqrt{\left(\frac{r}{2p}\right)^2 + \frac{r}{p}}$ such that $x_1 < 1 < x_2$. Note, that $x_1 x_2 = 1$.

Also:

$$V_{K+1} - V_K = c \tag{33}$$
$$V_{-K} - V_{-K-1} = b \tag{34}$$

Solving for the constants C and D:

$$C = \frac{\left(b - cx_1^{2K+1}\right) x_2^K}{\left(x_1 - 1\right)\left(x_2^{2K+1} - x_1^{2K+1}\right)} \tag{35}$$
$$D = -\frac{\left(b - cx_2^{2K+1}\right) x_1^K}{\left(x_2 - 1\right)\left(x_2^{2K+1} - x_1^{2K+1}\right)} \tag{36}$$

Incentive compatibility requires that $V_{k+1} - V_k > c$ for $-K < k < K$. This implies that

$$G\left(k\right) = Cx_1^k\left(x_1 - 1\right) + Dx_2^k\left(x_2 - 1\right) > c \tag{37}$$

This expression can be simplified by using $y = x_1^{k-K}$ (note, that $1 \le y \le x_1^{-2K}$)):

$$G\left(y\right) = \frac{1}{x_1^{-2K-1} - x_1^{2K+1}}\left[y\left(b - cx_1^{2K+1}\right) + \frac{1}{y}\left(cx_1^{-2K-1} - b\right)\right] \tag{38}$$

The derivative of G with respect to y can be calculated:

$$G'(y) = \frac{1}{x_1^{-2K-1} - x_1^{2K+1}} \left[\left(b - cx_1^{2K+1}\right) - \frac{1}{y^2}\left(cx_1^{-2K-1} - b\right)\right] \tag{39}$$

If $cx_1^{-2K-1} - b < 0$ our solution is valid because $G(k)$ is decreasing over the entire range $-K < k < K$ which ensures IC. However, if $cx_1^{-2K-1} - b > 0$ this is not necessarily true any longer. Now G is minimized at $y^* = \sqrt{\frac{cx_1^{-2K-1}-b}{b-cx_1^{2K+1}}}$. Clearly, $y^* \approx \sqrt{\frac{c}{b}}x_1^{-K}$ as $K \to \infty$. Therefore, there for sufficiently large K the minimum of $G(k)$ will be reached within the permissible range. Then

$$G(y^*) = \frac{\sqrt{bc\left(x_1^{-2K-1} + x_1^{2K+1}\right) - b^2 - c^2}}{x_1^{-2K-1} - x_1^{2K+1}} \tag{40}$$

Rewriting the above expression for $z = x_1^{2K+1}$ (note, that $0 < z < 1$):

$$G(y^*) = \frac{\sqrt{bc\left(z + \frac{1}{z}\right) - b^2 - c^2}}{\frac{1}{z} - z} < c \tag{41}$$

We now show that

$$H(z) = bc\left(z + \frac{1}{z}\right) - b^2 - c^2 - c^2\left(\frac{1}{z} - z\right)^2 \tag{42}$$

is increasing in z.

$$H'(z) = \left(1 - \frac{1}{z^2}\right)\left[bc - 2zc^2\left(1 + \frac{1}{z^2}\right)\right] \tag{43}$$

Note, that $\frac{c}{z} > b$ by assumption. Hence:

$$H'(z) > \left(\frac{1}{z^2} - 1\right)\left[2\frac{c^2}{z} - bc\right] > \left(\frac{1}{z^2} - 1\right)[2bc - bc] > 0 \tag{44}$$

But, $H(1) = 2bc - b^2 - c^2 < 0$. Therefore, whenever y^* is in the permissible range there cannot be an equilibrium.

Therefore, for existence of equilibrium we have to check two cases (a) $b > \frac{c}{z}$, and (b) $b < \frac{c}{z}$ and $y^* < 1$. The last condition can be simplified to $b > c\frac{z+\frac{1}{z}}{2}$. Note, that

$$c\frac{z + \frac{1}{z}}{2} = \frac{c}{z}\frac{1 + z^2}{2} < \frac{c}{z} \tag{45}$$

because $0 < z < 1$.

Hence, an equilibrium will exist if and only if $b > c\frac{\frac{1}{z}+z}{2}$. This allows for definition of a cutoff value z^* for the existence of equilibrium:

$$z^* = \frac{b}{c} - \sqrt{\left(\frac{b}{c}\right)^2 - 1} \tag{46}$$

Equilibrium will exist if and only if $z > z^*$. QED