The Formation of Ghettos as a Local Interaction Phenomenon

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Abstract

I analyze a simple evolutionary model of residential segregation based on decentralized racism which extends Schelling's (1972) well-known tipping model by allowing for local interaction between residents. The richer set-up explains not only the persistence of ghettos, but also provides a mechanism for the rapid transition from an all-white to an all-black equilibrium.

On one-dimensional *streets* segregation arises once a group becomes sufficiently dominant in the housing market. However, the resulting ghettos are not persistent, and periodic shifts in the market can give rise to "avenue waves". On two-dimensional *inner-cities*, on the other hand, ghettos can be persistent due to the "encircling phenomenon" if the majority ethnic group is sufficiently less tolerant than the minority. I review the history of residential segregation in the US and argue that my model can explain the rapid rise of almost exclusively black ghettos at the beginning of the 20th century.

For the analysis of my model I introduce a new technique to characterize the medium and long-run stochastic dynamics. I show that *clustering* predicts the behavior of large-scale processes with many agents more accurately than standard stochastic stability analysis, because the latter concept overemphasizes the 'noisy' part of the stochastic dynamics.

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1 Introduction

Residential segregation along ethnic and racial lines is a fact of life in the US and in many other countries. There is a substantial body of literature which documents the economic and social costs of segregation. Ethnic sorting retards inter-generational improvements for relatively disadvantaged groups[1], reduces empathetic connections between ethnic groups and therefore diminishes political support for redistribution (as in Cutler, Elmendorf and Zeckhauser (1993)) and increases statistical discrimination because whites, for example, end up relying more on stereotypes of blacks instead of actual experience (Wilson 1987).

While we know a great deal about the outcomes of residential segregation on society the mechanism which gives rise to ghettos[2] in the first place is much less understood. This paper provides a simple theory of segregation based on decentralized racism which can explain the key empirical facts. First, ghettos historically developed fairly rapidly: in the US the core of the black ghettos formed between 1900 and 1920. Second, black ghettos in particular tend to be very persistent over time once they cover a large contiguous geographical area, such as Harlem in New York City.[3] Ghettos are far less stable, on the other hand, if they extend only over a single street as examples from Chicago's avenues at the turn of the century show. My model abstracts away from other contributing factors to segregation, such as sorting by socio-economic differences and collective-action racism. I argue in the empirical part of my paper that these effects on their own cannot explain the key empirical facts about segregation in the US.

Existing models of segregation are typically variants of Schelling's (1972) influential tipping model which can generate multiple stable segregation equilibria.[4] But like most models with multiple equilibria, the basic tipping model suggests no

[1]Borjas (1995) found that ethnicity has an external effect even after controlling for parental background and the socio-economic characteristics of a neighborhood. Neighborhood peers appear to affect the skills and norms of the young in particular, such as the probability of being involved in crime and the propensity of youths to be out of school or work (see, for example, Case and Katz (1991), and Glaeser, Sacerdote, and Scheinkman (1996)). Cutler and Glaeser (1997) compared the outcomes of blacks between cities and found that blacks in racially more segregated cities earn less income and are more likely to become single mothers or drop out of high school.

[2]The term "ghetto" is used nonpejoratively throughout the paper in order to denote a racially or ethnically segregated community.

[3]Cutler, Glaeser and Vigdor (1997) documented that the correlation across cities between segregation in 1890 and segregation in 1990 is as high as 50 percent. Residential segregation affected all ethnic minorities in the US to varying degrees. For African Americans, however, it is unique in its severity and persistence over several generations. Second- and third-generation non-black immigrants generally lived in much less segregated neighborhoods than their parents. For a good reference see chapter three in K. Taeuber and A. Taeuber (1965).

[4]For example, Galster (1990) and Cutler, Glaeser and Vigdor (1997) use variants of the tipping model for their empirical studies.

mechanism for moving between an all-white equilibrium and a ghetto equilibrium. The theory can therefore explain the persistence but not the formation of ghettos. Perhaps surprisingly, these limitations can be overcome by allowing for a richer (and more natural) geometry of interaction between residents where agents care more about neighbors who are geographically close to their apartment than about residents further away.[5]

My model analyzes the location decision of two ethnic groups, which I refer to as 'blacks' and 'whites' for convenience. The model is described by four basic parameters: the tolerance levels of both groups α_b and α_w, their balance λ in the housing market and the geometry G of neighborhood interaction. The geometries I mainly consider are one-dimensional *streets* and two-dimensional *inner-cities* where residents only care about their direct neighbors. For completeness I also look at *bounded neighborhoods* where each resident is neighbor to every other resident in the area such that my model reduces to a variant of Schelling's tipping model.

Residents leave the residential area randomly at a fixed rate and are replaced by newcomers from the housing market where a share λ of apartment seekers are white. Most newcomers, however, exhibit mild ethnic preferences and only consider an apartment if they do not feel 'isolated', i.e. at most a share $\frac{1}{2} < \alpha < 1$ of their prospective neighbors is of a different ethnicity. A small share ϵ of newcomers are completely tolerant and do not care about the ethnicity of their neighbors. These non-discriminating residents provide the 'noise' which is necessary to move the process from one equilibrium to the next.

On streets the relative strength of ethnic preferences alone cannot give rise to segregation. Once fear of isolation interacts with the balance in the housing market, however, streets can rapidly turn into ghettos. A sudden rise in the share of blacks in the market can transform the residential area because blacks can 'invade' the street around small clusters of black residents which have formed on the street by chance. The 'contagious' dynamics of this transformation is reminiscent of the model studied by Glenn Ellison (1993), who demonstrated how local interaction can speed up convergence to the long-run equilibrium. My model highlights the importance of the increase in the share of African Americans in the housing market for the rise of black ghettos. This situation occurred at the beginning of the 20th century when African Americans started to migrate from the rural South to the booming industrial centers of the North.

Segregation on streets, however, is not persistent because the forces that give rise to ghettos are symmetric: as soon as whites dominate the housing market again, a ghetto can equally rapidly disintegrate. Temporary imbalances in the housing market can therefore give rise to periodic transformations of ethnic neighborhoods or "avenue waves". In the US there existed a natural source of variation

[5] Incidentally Schelling himself sketched a two-dimensional model of segregation in his book *Micromotives and Macrobehavior* (1978).

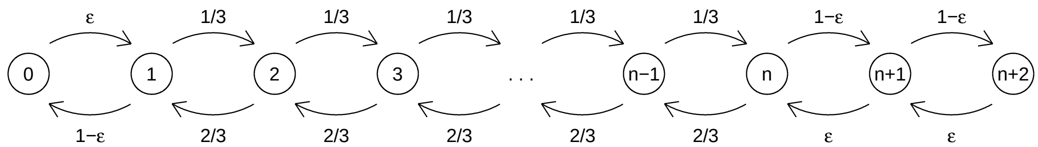


Figure 1: Illustration of the dynamics of a continuous-time random walk over the set of states $\{0, 1, .., n + 2\}$

in the housing market balance for non-black minorities at the turn of the century because large waves of immigrants from certain ethnic groups, such as Southern Europeans, Scandinavians or Russians, entered the country at different points in time. In section 5 I present evidence from Chicago which documents the occurrence of "avenue waves" along the main arteries as predicted by my model.

Ghettos can exhibit pronounced persistence, however, once the process evolves in a two-dimensional inner-city area. As long as blacks are sufficiently more tolerant than whites the process will again give rise to rapid segregation as soon as blacks dominate the housing market sufficiently. The resulting ghettos, however, will not disintegrate after subsequent changes in the balance of the housing market. The reason for this stasis is the "encircling phenomenon": in two dimensions, randomly forming clusters of white residents cannot expand as easily as on streets because white residents at the convex boundary of the cluster have on average more black than white neighbors. This prevents the kind of contagious dynamics through which white clusters on streets can break up a black ghetto. Inner-city areas therefore behave like hybrids: they can be transformed into ghettos (like streets) which are subsequently stable (like the all-black equilibrium of Schelling's tipping model).

For the analysis of the medium and long-run behavior of my model I develop a new technique to characterize the stochastic evolution of large-scale dynamical systems, which I believe can be fruitfully applied to both existing and future models. The standard technique for understanding the long-run behavior of stochastic dynamics in evolutionary game theory has been stochastic stability analysis, which was introduced in the seminal work by Young (1993) and Kandori, Mailath and Rob (1993) and recently extended by Ellison (1999). The model of this paper, however, illustrates that stochastic stability can seriously mispredict the behavior of a process with many agents. Incidentally, these are exactly the environments where evolutionary reasoning seems most adequate.

The shortcomings of stochastic stability can be most easily explained with the help of a simple example. Figure 1 illustrates the dynamics of a random walk on the integers $\{0, 1, .., n + 2\}$ in continuous time. Between states 1 and n the process evolves according to the 'undisturbed' dynamics, while between states 0 and 1 and between states n and $n+2$ the process is governed by 'noisy' dynamics. For any fixed size n only the state $n+2$ is stochastically stable in the sense that the process spends almost all its time at $n+2$ as the noise term ϵ becomes small. To see the intuition for this result, note that it takes two mutations to leave the basin of attraction of state $n+2$, but just one mutation to leave the basin of attraction of state 0.[6] This reasoning, however, does not take into account the nature of the undisturbed dynamics at all, even through it pushes the process away from the stochastically stable state.

Stochastic stability can thus lead us to mispredict the long-run behavior of the process. Assume that we want to find the process inside a small δ-neighborhood $[(1-\delta)\,n, n+2]$ of the stochastically stable state with probability $\gamma > 0$. It can be shown that the noise term ϵ then has to be smaller than $\left(\frac{3}{2}\right)^{-n}$. But as the size of the system increases stochastic stability will capture the dynamics adequately only for extremely small noise. Even more worrisome, the waiting time to reach the stable state $n+2$ becomes unrealistically large for such small ϵ.

Stochastic stability analysis therefore tends to work best for small-scale stochastic systems, while most evolutionary environments, such as residential neighborhoods, involve many interacting agents. For the preceding example one can in fact demonstrate that for a fixed noise term ϵ the process will spend almost all its time close to the state 0 as the size n of the system increases. The long-run evolution of the system is completely determined by the biased undisturbed dynamics of the process.

This insight immediately suggests an alternative technique to characterize stochastic dynamics. *Clustering* looks at the dynamics of a system as its size n increases and therefore takes into account both the disturbed and the undisturbed dynamics of a model. This makes clustering a more robust equilibrium concept than stochastic stability for systems with many agents.

The balance of the paper is organized as follows. In the next section I lay out a general model for a residential segregation process and introduce the notion of clustering. The new technique is then applied to streets in section 3 and to inner-cities in section 4, in order to characterize the long-run equilibria of the model and to find bounds on the waiting time to reach those equilibria. In section 5 I discuss how my theory can help us to understand the formation of ghettos at the beginning of the 20th century in the US. I also present evidence of "avenue waves" from Chicago and describe in detail Harlem's transformation from a white upper-class neighborhood into a black ghetto between 1900 and 1930. The relationship between stochastic stability and clustering is explored in section 6 using the waiting time terminology introduced by Ellison (1999). In order to demonstrate the usefulness

[6] The basin of attraction is defined with respect to the undisturbed dynamics, i.e. $\epsilon = 0$.

of clustering as a general technique I revisit a well-known application of stochastic stability by Ellison (1993) and illustrate how clustering can make the predictions of Ellison's paper robust to changes in the dynamics.

2 A Framework for Analyzing Segregation

This section introduces a simple evolutionary model of segregation and the notion of clustering which will be used to analyze the medium and long-run behavior of the resulting Markov process. In the case of bounded neighborhoods my model reduces to Schelling's (1972) tipping model, and I demonstrate why this setup describes the dynamics of segregation insufficiently. Although the tipping model allows for both an all-white and an all-black ghetto equilibrium, the process remains locked into basins of attraction around those equilibria. The medium-run dynamics is solely determined by the initial conditions, and there is no mechanism which gives rise to ghettos within a realistic time frame.

2.1 The Basic Setup

A residential area of size n consists of n residents $R = \{z_1, z_2, ..., z_n\}$. They form the vertices of a connected graph G defined through a symmetric neighborhood relation $G \subset R \times R$.[7] Each resident z has a natural neighborhood $N(z) = \{z' \,|(z, z') \in G\}$. I restrict attention to three possible residential geometries. *Bounded neighborhoods* $G_B(n)$ have a complete graph G such that individual neighborhoods coincide with the entire residential area. There are also two local geometries with easy intuitive representations: one-dimensional *streets* $G_S(n)$ and two-dimensional *inner-city areas* $G_C(n)$. On a street, residents are located on a circle with each agent having two neighbors on both sides. An inner-city area consists of a torus of size $\sqrt{n} \times \sqrt{n}$ such that each resident has four neighbors. Streets have the lowest possible connectivity of a regular connected graph, while bounded neighborhoods have the highest. Inner-cities take an intermediate position.[8] Both streets and inner-cities have intuitively related graphs of higher order if we allow for individual neighborhoods of radius $r > 1$ in the standard Euclidean norm. These geometries are denoted with $G_S^r(n)$ and $G_C^r(n)$ respectively. Figure 2 shows both a street and an

[7] The graph is connected if any two residents are connected through a transitive chain of neighborhood relationships.

[8] The *connectivity* $C(G)$ of a finite graph G is defined as the lowest upper bound for the minimum length path connecting any two residents on the graph, i.e. $C(G) = \max_{z_i, z_j \in R} d(z_i, z_j)$ where $d(z_i, z_j)$ denotes the length of the minimum length path connecting z_i and z_j. The smaller $C(G)$ the better connected the graph. Bounded neighborhoods have $C(G_B) = 1$, a street $G_S(n)$ has connectivity $\left[\frac{n+1}{2}\right]$ and the inner city $G_C(n)$ has an intermediate connectivity of $2\left[\frac{\sqrt{n}}{2}\right]$.

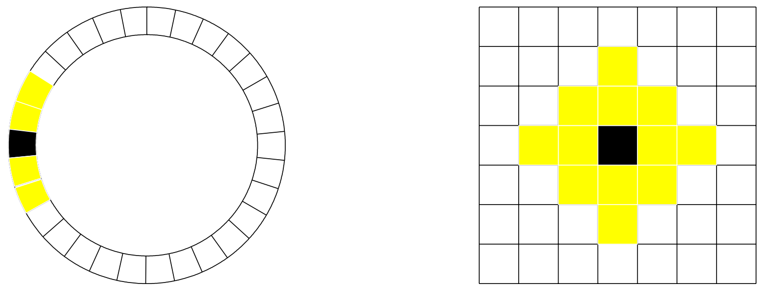


Figure 2: Street and inner-city geometries with individual neighborhoods of radius 2

inner-city with respective neighborhoods of radius 2.[9] Note that for a large radius r the individual neighborhood of a resident comprises the entire residential area, and one again obtains the bounded neighborhood geometry.

The residential area is populated by two ethnic groups whom I refer to as 'blacks' and 'whites' throughout the paper. At each point in time the pattern of settlement is defined by a configuration $\eta : G_i \rightarrow \{0, 1\}$, where the values 0 and 1 denote a white and black resident respectively.[10] The residential area allows a total of 2^n configurations which form the configuration set Z. A cluster in some configuration η is defined as a connected set of residents of the same ethnic group.[11]

All residents in the area are assumed to have an area-specific socio-economic status. Time is continuous, and agents get 'lucky' according to an i.i.d. Poisson process at rate 1. 'Lucky' agents immediately become a member of the next highest socio-economic group and move out of the area because they can afford better housing. Vacant flats are occupied by newcomers from a large pool of prospective tenants. A share λ of those is white and a share $1 - \lambda$ is black.

A (small) share ϵ of prospective tenants are completely tolerant in the sense that they do not care about the ethnic composition of their individual neighborhoods. The remaining share $1 - \epsilon$ of prospective tenants have mild ethnic preferences because they are afraid of isolation at their new apartments. All whites and all blacks have identical, group-specific tolerance levels α_w and α_b respectively. The tolerance level marks the maximum share of neighbors of a different ethnicity a

[9]Looking at a circle and a torus respectively avoids the need to specify boundary conditions. All results hold for open linear streets and rectangular inner-cities, as well, once the decision rules for residents at the boundary are suitably adapted.

[10]I generally treat a lattice cell as a single resident. For densely populated cities such as New York City, however, it might be more adequate to interpret lattice cells as entire apartment blocks, as the owners of these buildings usually did not mix tenants of different racial groups.

[11]Connectedness is defined with respect to simple streets and inner-city geometries with radius of interaction $r = 1$.

prospective tenant is prepared to accept. I assume that $\frac{1}{2} \leq \alpha_i < 1$, i.e. agents are generally happy to live in integrated areas where both ethnic groups share the neighborhood equally.[12] I assume throughout the paper that the minority group ('blacks') is more tolerant than the majority group ('whites'), i.e. $\alpha_w \leq \alpha_b$.[13]

The housing market operates as follows. All prospective tenants have a basic willingness to pay WTP, which depends only on their socio-economic status and is equal for both whites and blacks. If a resident feels isolated, however, her willingness to pay decreases to $WTP - D$ for some $D > 0$. An apartment is then allocated amongst the highest bidders through randomization. One can derive a switching function which denotes the probability that the color of a tenant switches conditional on the previous tenant having moved out. I denote the share of black neighbors of a resident z in configuration η with $x(\eta, z)$ and the share of white neighbors with $y(\eta, z)$. The probability of a color switch g_w if the previous tenant was white then becomes:

$$g_w^\epsilon (x(\eta, z)) = \begin{cases} \frac{(1-\lambda)\epsilon}{(1-\lambda)\epsilon+\lambda} & \text{for } x < 1-\alpha_b \\ 1-\lambda & \text{for } 1-\alpha_b \leq x \leq \alpha_w \\ \frac{1-\lambda}{1-\lambda+\epsilon\lambda} & \text{for } x > \alpha_w \end{cases} \tag{1}$$

Analogously, the probability g_b for a switch from a black to a white tenant is:

$$g_b^\epsilon (y(\eta, z)) = \begin{cases} \frac{\lambda\epsilon}{\lambda\epsilon+1-\lambda} & \text{for } y < 1-\alpha_w \\ \lambda & \text{for } 1-\alpha_w \leq y \leq \alpha_b \\ \frac{\lambda}{\lambda+\epsilon(1-\lambda)} & \text{for } y > \alpha_b \end{cases} \tag{2}$$

Figure 3 illustrates the typical shape of the resulting switching functions in terms of the share of neighbors of the opposite color.

The evolution of the residential neighborhood can now be described by a continuous time Markov chain η_t on the space of configurations Z where η_0 is the initial configuration which is set by some historical accident.

Remarks on the Setup: 1. Which of the two local geometries approximates real-life residential areas best? It is natural to think of geographic entities, such as residential neighborhoods, in a two-dimensional setting. On the other hand, 'streets' might capture the neighborhood interaction on large avenues more appropriately.[14]

[12] If $\alpha_i < \frac{1}{2}$ for both groups, segregation would be the socially efficient outcome.

[13] Empirical studies such as the General Society Survey reveal that whites discriminate more strongly than blacks (see Cutler, Glaeser and Vigdor (1997)).

[14] Residents on such major roads certainly had some preferences concerning the racial composition of side streets, but they presumably put greater weight on the residents living along the avenue: a majority of shops, public transport and institutions such as churches would be located along the avenues, making social interaction with residents there more likely.

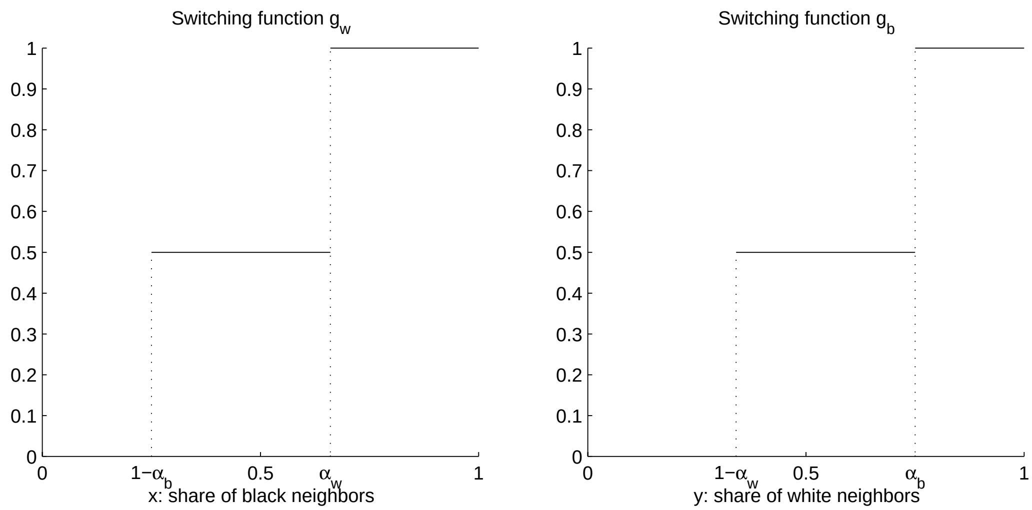


Figure 3: Switching functions for white/ black transition (g_w) and black/ white transition (g_b) for tolerance levels $\alpha_w = \frac{2}{3}$, $\alpha_b = \frac{3}{4}$, share of completely tolerant agents at $\epsilon = 0$ and the share of whites in society at $\lambda = 0.5$

2. Prospective tenants in my model behave only in a boundedly rational manner and just take the contemporaneous ethnic balance of a neighborhood into account. Forward-looking rational agents with a positive discount factor should anticipate the probabilistic evolution of a residential area and possibly take into account additional available information such as the total ethnic balance of the area they move into. The computational requirements on prospective tenants become enormous, however, even for moderately large residential areas. I therefore adopt the myopia hypothesis which is commonly employed in evolutionary game theory (see, for example, Kandori, Mailath and Rob (1993) and Young (1993)) and allows me to concentrate my analysis on the dynamics of segregation.

3. Fear of isolation gives rise to an S-shaped frequency distribution of tolerance levels within each ethnic group. The empirical evidence suggests that the tolerance distribution is indeed highly non-linear and S-shaped (see Galster (1990)). The results of this paper carry through for more general tolerance distributions and richer models of the housing market as long as the tails of the reduced form switching functions are flat, i.e. whites (blacks) will mainly seek out white (black) neighborhoods.

4. Discrimination in this model operates only through destination selectivity of prospective tenants in the housing market. Schelling's (1972) original tipping model also allows agents to leave a neighborhood at an increased rate if they feel

isolated.[15] This second channel can be easily incorporated in my model without changing the qualitative predictions. Destination selectivity, however, seems to be the more important channel, as moving costs are presumably higher than the search cost which is associated with excluding some apartments from further consideration. Furthermore, my main application of the model concerns the formation of ghettos at the turn of the century when city growth was rapid and the residential turnover rate was high.

5. The model can be regarded as a partial equilibrium building block for a richer general equilibrium model of a growing city consisting of many residential areas of different socio-economic status. Historically, the frantic expansion of Northern cities in the US at the turn of the century was accompanied by a chain of succession and invasion. As wealthy middle class citizens in New York or Chicago gradually abandoned the city for the new suburbs, they were replaced by successful immigrants who had left behind their lower class origins. Their place, in return, was occupied by new immigrants and migrants from rural areas.

2.2 Characterizing the Stochastic Dynamics through Clustering

It is easy to see that the model has a unique ergodic distribution μ_∞^ϵ over the set of configurations Z which describes the long-run behavior of the system.[16] Therefore the long-run behavior of the system is independent of the initial conditions. But this observation is of little interest unless we find a way to classify the ergodic distribution. Will the process spend most of its time around segregation configurations or around mixed configurations? How does the equilibrium depend on parameters of the model, i.e. the geometry, the tolerance levels of both groups and the balance in the housing market?

The standard technique for classifying the ergodic distribution μ_∞^ϵ is stochastic stability analysis, which was developed by Young (1993) and Kandori, Mailath and Rob (1993). All results from this literature can be applied to my model if the small share ϵ of completely tolerant residents is interpreted as 'noise'. A configuration η is then called *stochastically stable* if $\lim_{\epsilon\to 0}\mu_\infty^\epsilon > 0$ for some fixed geometry $G_i(n)$ $(i = B, S, C)$.

I will demonstrate in section 6 that stochastic stability explains the long-run dynamics of the residential neighborhood process very poorly for large-scale res-

[15] In his sketch of a local tipping model Schelling (1978) omits destination selectivity altogether.

[16] Appendix A shows how to associate a Markov chain with the continuous time Markov process. The transition matrix P^ϵ of that chain is regular as $(P^\epsilon)^n$ has no non-zero entries - each configuration of the geometry can be reached after n steps with positive probability (Kemeny and Snell 1960, Theorem 4.1.2). Therefore the process is ergodic (Kemeny and Snell 1960, Theorem 4.1.4).

idential neighborhoods. The intuition for this failure will be the same as for the example I gave in the introduction. Stochastic stability describes the process well only for extremely small ϵ. This requirement is troubling because the share of tolerant residents might be low but is certainly not negligible. Even more worrisome is the effect that such a small 'noise' parameter has on the waiting time before the stochastically stable configuration is reached for the first time. For large neighborhoods, convergence will be so unrealistically slow that the analysis will tell us nothing about the medium-run behavior of the process.

This insight quite naturally suggests an alternative to stochastic stability which fixes the noise term ϵ and instead considers very large residential areas, i.e. lets $n \to \infty$.[17] As the main parameter of interest is the ethnic balance in the residential area, I formally define the concept of *clustering* for the long-run share $\tilde{X}_n^\epsilon$ of black residents which is a scalar random variable.

Definition 1 *A sequence of random variables* $\left\{\tilde{X}_n\right\}$ *on the interval* $[0,1]$ *is said to cluster over the set* $I \subset [0,1]$ *if* $P\left(\tilde{X}_n \in I\right) \to 1$ *as* $n \to \infty$.

For example, we can interpret clustering of the residential neighborhood process on a street $G_S(n)$ around a black share close to 1 in the sense that large streets will become black ghettos in the long run.

Although clustering captures the long-run behavior of the process well, it does not tell us how fast a neighborhood I is reached over which the process clusters. Waiting times are a very useful measure for the speed of convergence to equilibrium, as was first emphasized by Ellison (1993) in the context of stochastic stability. With respect to clustering, the relevant measure is the maximum waiting time $W(n, I)$ in which the process reaches I for the first time starting from any initial configuration:

$$W(n, I) = \max_{\zeta \in Z} \left[E\left(\min t \,|X(\eta_t) \in I \quad \text{and} \quad \eta_0 = \zeta\,\right)\right] \tag{3}$$

$X(\eta)$ here denotes the share of black residents in configuration η. Unless that waiting time remains bounded as n increases, the evolution of the process will be determined by the initial conditions rather than the long-run equilibrium.

2.3 Schelling's Tipping Model as a Benchmark

It is instructive to start the analysis of the residential neighborhood process for bounded neighborhoods $G_B(n)$ because the model becomes a variant of Schelling's (1972) well-known tipping model. The entire intuition for the behavior of the

[17] The type of geometry is assumed to be fixed when taking the limit.

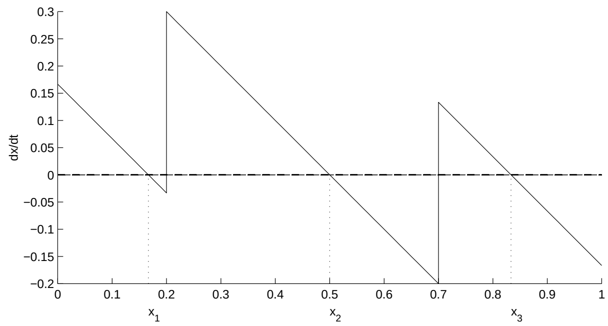


Figure 4: Graph of $\frac{dx}{dt}$ in the deterministic approximation to the residential neighborhood process on $G_B(n)$ for large n ($\alpha_b = 0.8$, $\alpha_w = 0.7$, $\lambda = 0.5$, $\epsilon = 0.2$)

process on large bounded neighborhoods can be derived from the deterministic approximation of the change in the share of black residents $x(t)$:

$$\frac{dx}{dt} = (1-x)\, g_w^{\epsilon}(x) - x g_b^{\epsilon}(1-x) \tag{4}$$

Figure 4 shows the graph of $\frac{dx}{dt}$ and illustrates that the deterministic approximation of the process has, in general, multiple stable steady state equilibria: two segregation equilibria $x_1 = \frac{(1-\lambda)\epsilon}{(1-\lambda)\epsilon+\lambda}$ and $x_3 = \frac{1-\lambda}{1-\lambda+\lambda\epsilon}$, and possibly one integrated equilibrium $x_2 = 1 - \lambda$.[18] The corresponding basins of attraction are $B_1 = [0, 1-\alpha_b)$,$B_2 = (1-\alpha_b, \alpha_w)$, and $B_3 = (\alpha_w, 1]$, respectively.

The evolution of the deterministic approximation to the process is therefore entirely determined by the initial conditions, which is a highly unsatisfactory feature of bounded neighborhoods. The choice of the initial share of black residents x_0 is indeterminate without making arbitrary assumptions about the history of the process. While the model does well in explaining the persistence of ghettos, it suggests no mechanism for moving between the steady states.

The intuition which we gained from the deterministic approximation continues to hold for the stochastic model, as the next theorem shows. The stochastic drift will select one of the steady states in the long run, depending on the parameter values. For simplicity I restrict attention to the most interesting case where blacks dominate the housing market. In this case the residential area will turn into a black ghetto, and the process clusters around x_3 unless there are too many completely tolerant agents in the housing market such that the process clusters around the

[18] I assume that the share of completely tolerant residents ϵ is sufficiently small such that $x_1 < 1-\alpha_b$ and $x_3 > \alpha_w$. The integrated equilibrium might not exist if the housing market is sufficiently unbalanced, i.e. $\lambda > \alpha_b$ or $\lambda < 1-\alpha_w$.

Table 1: Comparison of waiting times $W(n, I)$ for reaching the neighborhood $I = [0.97, 1]$ of the ghetto steady state $x_3 = 0.99$ ($\alpha_b = \frac{3}{4}$, $\alpha_w = \frac{2}{3}$, $\epsilon = 0.05$, $\lambda = 0.2$)

Size of area	n=100	n=200	n=300	n=400
Waiting time $W(n, I)$	21	137	627	4908

Estimated standard errors for the waiting times are 10% or less.

integrated steady state $x_2 = 1 - \lambda$. However, this characterization of the long-run behavior of the process is meaningless for its medium-run evolution. The process is tightly 'locked' in the basins of attraction around the three steady states, as the second part of the theorem shows. For all practical purposes the behavior of the process is indeed determined by the initial conditions. Therefore, the tipping model can not explain the formation of ghettos.

Theorem 1 *Consider a residential neighborhood process on $G_B(n)$ with initial share of black residents x_0 in one the three basins of attraction B_i ($i = 1, 2, 3$). Blacks dominate the housing market, i.e. $\lambda < \frac{1}{2}$.*

1. *(Long-run behavior) The process clusters around any neighborhood of the ghetto steady state x_3 if $\frac{\epsilon^{1-\alpha_w}}{1-\lambda+\lambda\epsilon} < 1$, i.e. the share of completely tolerant apartment seekers is sufficiently small (and x_2 exists). Otherwise, the process clusters around the integrated steady state $x_2 = 1 - \lambda$.*

2. *(Medium-run behavior) The process reaches a δ-neighborhood of the steady state x_i before it can leave the basin of attraction with probability approaching 1 as $n \to \infty$. The conditional waiting time for this event is bounded above by some finite W_δ. Moreover, the waiting time to reach a neighborhood of the steady state chosen in the long-run is of the order $A(x_0)^n$ where $A(x_0) > 1$ if the process starts outside the basin of attraction of that steady state.*[19]

Proof: see appendix C

Example: A little numerical example illustrates the irrelevance of the long-run equilibrium for the medium-run behavior of the process. I consider the case where the black and white tolerance levels are $\alpha_b = \frac{3}{4}$ and $\alpha_w = \frac{2}{3}$ respectively, and 5 percent of all agents are completely tolerant ($\epsilon = 0.05$). I assume that

[19] I assume that $1 - \alpha_w < \lambda < \alpha_b$ such that a bounded neighborhood can stay integrated around x_2 in the medium run. If $\lambda < 1 - \alpha_w$ the process will reach its long-run equilibrium in finite time for $x_0 > 1 - \alpha_b$. If $\lambda > \alpha_b$ the process will reach its long-run equilibrium in finite time only for $x_0 > \alpha_w$.

the neighborhood has initially been in an all-white steady state ($\lambda = 1$) when an influx of blacks into the housing market occurs, causing the share of whites in the market to fall to 20 percent. The all white steady state and the all black steady states are $x_1 = 0.17$ and $x_3 = 0.99$ respectively; there is no integrated steady state. Theorem 1 tells us that under these circumstances the process clusters around any neighborhood I of the ghetto steady state, say $I = [0.97, 1]$. How long will it take until the bounded neighborhood has turned into a ghetto? Table 1 shows the results from a simulation for neighborhoods of various sizes.[20] The data nicely confirm the theory, as the waiting times increase rapidly with the size n of the bounded neighborhood. In the medium run, the behavior of the process on even moderately large residential areas is therefore entirely determined by the fact that the process started from an all-white configuration. As a response to the dominance of blacks in the housing market their share in the area will increase rapidly to about 17 percent, i.e. the steady state value x_1. The process will then oscillate around this *meta equilibrium*, but is unlikely to escape its basin of attraction within any realistic time frame.

3 Rapid Segregation and "Avenue Waves" on Streets

The previous section demonstrated that the original tipping model exhibits stasis around its steady states. It is noteworthy that the persistence of the all-white and all-black segregation steady states is preference based: the apartment seekers of the minority group feel isolated and avoid the residential area. In particular, an increased presence of blacks in the housing market does not trigger the transformation of the area into a ghetto in the medium run.

On streets, on the other hand, the residential neighborhood process behaves in a radically different manner because it gives a role to the balance in the housing market. Black (white) segregation on streets is upheld in the long run because the share of blacks (whites) in the housing market exceeds a critical level. Moreover, this mechanism lets the process reach its long-run equilibrium rapidly. In contrast to the standard tipping model, streets do not behave differently in the medium run and in the long run. Streets, therefore, provide a mechanism for moving between all-white and all-black equilibria through changes in the composition of the housing market.

I begin my analysis with a heuristic argument in order to illustrate why streets become ghettos in the long run if blacks sufficiently dominate the housing market. The intuition is cleanest for the case where the tolerance levels are close to $\frac{1}{2}$. For

[20] Note that the maximum waiting time $W(n, I)$ coincides in this case with the waiting time of reaching I starting from the all-white configuration.

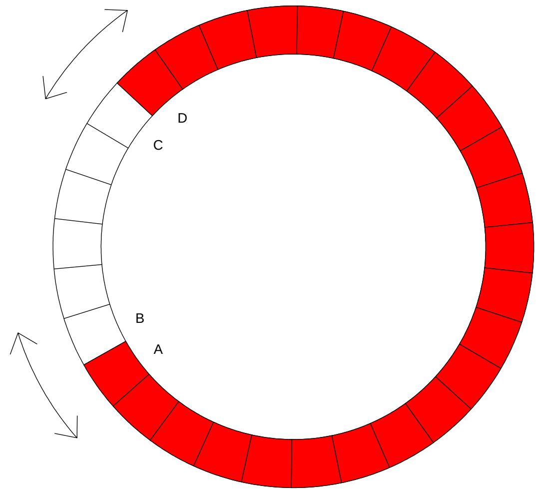


Figure 5: Black ghetto on a street $G_S^{r=1}(n)$ with a single white cluster: if residents A to D move, members of both racial groups are equally interested in vacant apartments.

simplicity I only look at simple streets $G_S^{r=1}(n)$, although the argument is readily generalized to higher-order streets. A ghetto on such a street will occasionally face invasion by small white clusters of completely tolerant apartment seekers, as illustrated in figure 5. Under the assumption that the share ϵ of tolerant agents is small, vacant apartments *inside* of black and white clusters are almost always taken only by black and white residents respectively due to the assumption on the tolerance levels. However, if apartments at the boundary of the cluster become vacant (such as A or D) apartment seekers from both ethnic groups will be interested in them. The boundaries of the black cluster therefore move according to a random walk with absorption (the process ends if one of the clusters vanishes). The drift of this random walk is solely determined by the composition of the housing market, i.e. λ. As blacks dominate the housing market the white cluster is likely to shrink rather than grow.

We can now ask the question, what would happen to this cluster if the circle was infinite and it could not interact with other random white clusters? Standard theory tells us that the cluster would die out with probability 1, and its expected maximum length would be finite and determined by the negative drift only.[21] From this observation we can conclude that white clusters form and die independently from one another to a first approximation, as long as the share ϵ of completely tolerant agents is small and the size n of the street is large. Therefore, the equi-

[21]These are standard results from random walk theory (Stirzaker 1994, section 5.6).

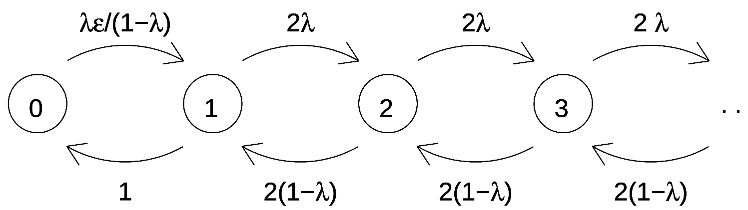


Figure 6: Transition rates for the length of a white cluster originating at a single apartment z on the street $G_S^{r=1}(n)$

librium share of white residents in the ghetto can be derived by calculating the average length of white clusters which originate from some fixed apartment z on the street.[22] The length of such a cluster forms a random walk with transitions rates as indicated in figure 6. The probability x that the originating apartment is black then becomes:[23]

$$x = \left(1 + \epsilon \frac{\lambda(2\lambda+1)}{1-\lambda} + \epsilon \frac{2\lambda^2}{(1-2\lambda)(1-\lambda)}\right)^{-1} \tag{5}$$

From formula 5 one can immediately deduce that if the share of blacks in the housing market is greater than 50 percent and the share of completely tolerant agents is small the street will become a ghetto.

The next theorem makes this heuristic argument precise and generalizes it for the case where the tolerance levels are not necessarily close to $\frac{1}{2}$. If the share of whites in the market falls below some critical level $\hat{\lambda}$, the residential neighborhood process on a street will turn into a ghetto in the long run and cluster around a black share of $x = 1$. On the other hand, if the share of whites exceeds some critical level $\tilde{\lambda}$ the process clusters around the all-white equilibrium.

Theorem 2 *Given is a street $G_S^r(n)$ with group tolerance levels α_w and α_b. Then there exist critical values $0 < \hat{\lambda}(r, \alpha_w, \alpha_b) \leq \tilde{\lambda}(r, \alpha_w, \alpha_b) < 1$ such that the following holds.*

1. *If blacks sufficiently dominate the housing market ($\lambda < \hat{\lambda}$) the street becomes a black ghetto in the long run, i.e. the process clusters on the interval $[x_b^*(\epsilon), 1]$ with $\lim_{\epsilon \to 0} x_b^*(\epsilon) = 1$.*

2. *If whites sufficiently dominate the housing market ($\lambda > \tilde{\lambda}$) the street becomes a white ghetto in the long run, i.e. the process clusters on the interval $[0, x_w^*(\epsilon)]$ with $\lim_{\epsilon \to 0} x_w^*(\epsilon) = 0$.*

[22] I invoke the law of large numbers here.

[23] The expression for x is a first order approximation in ϵ.

Proof: see section 3.2

The description of the long-run equilibrium of the process is, of course, only relevant if it is reached reasonably quickly. Fortunately, this is the case on streets, as the next lemma shows. The intuition can be again derived for a simple street $G_S^{r=1}(n)$ where blacks dominate the housing market. I have demonstrated how white clusters tend to shrink rather than grow in such an environment. The argument can be flipped around in the sense that black clusters have to grow rather than shrink. Assume that the street is initially all-white. It is useful to divide the street up into k segments of some fixed size N. On each segment black clusters form after a waiting time of about $\frac{B}{N\epsilon}$. Such clusters can subsequently expand with positive drift. The problem is complicated by the fact that black clusters can be broken up by randomly forming white clusters. Ignoring this issue for the moment the black cluster would take over the neighborhood after some waiting time of the order AN (see, for example, lemma 5 in appendix B). The total waiting time until a segment becomes a ghetto is therefore of the order $\frac{B}{N\epsilon}+AN$. If the segments are large enough, each of them stays mostly black subsequently due to theorem 2. As $n \to \infty$ (i.e. $k \to \infty$) one can invoke a form of the central limit theorem in order to show that the waiting time $W(n, I)$ to reach some neighborhood I of the black ghetto equilibrium is of the order $O(1)$.

Lemma 1 *Consider a street $G_S^r(n)$ and assume that the share of whites in the housing market falls below the critical level $\hat{\lambda}$. If ϵ is sufficiently small the waiting time until the share of blacks reaches exceeds $1-\delta$ satisfies $W(n, [1-\delta, 1]) = O(1)$. An analogous result holds if the share of whites exceeds the critical level $\tilde{\lambda}$.*

Proof: see appendix F

It should be pointed out that the residential neighborhood process responds fast to changes in the composition of the housing market in *both* directions. While a street can quickly turn into a black ghetto, this development can reverse just as rapidly as soon as whites dominate the housing market again. In some sense, streets do too well in explaining the formation of black ghettos: if the composition of the housing market is highly volatile we should observe “avenue waves” instead of highly persistent ghettos. In section 5 I provide evidence for such waves in the case of Chicago. To summarize, while Schelling’s tipping model lacks a mechanism for ghetto formation but does well in terms of ghetto persistence the reverse is true for streets. One, therefore, would like a ’hybrid’ geometry between streets and bounded neighborhoods which can explain both phenomena. I argue in the next section that inner-cities provide such an environment.

The remainder of this section is devoted to some Monte Carlo simulations, which illustrate the fast response of streets to changes in the housing market, and to the proof of theorem 2. I discuss the steps of the proof in some detail because it introduces the *coupling* technique, a highly useful device for understanding Markov processes on a lattice.

3.1 Simulation Results on Critical Behavior and Speed of Adjustment

Theorem 2 and lemma 1 characterize the long-run and medium-run behavior of the residential neighborhood process on streets qualitatively. They do not permit us to numerically calculate the critical imbalance in the housing market which gives rise to ghettos, or the waiting time until convergence.[24] Simulations are therefore essential to assess the relevance of the theory. First, are the critical imbalances realistic, i.e. sufficiently bounded away from $\{0, 1\}$? Second, what does fast convergence in the medium run mean? Time is measured in my model in terms of tenant generations. Waiting times $W(n, I)$ of the order of 100, for example, translate into centuries when measured in real time.

The first round of simulations is aimed at finding the critical imbalance in the housing market such that a street becomes a ghetto in the long run. Figure 7 shows the long-run share of black residents depending on the share λ of whites in the housing market for streets with radius of interaction $r = 2$ and $r = 3$. The critical imbalance of the housing market is encouragingly close to 50 percent when both groups are equally tolerant such that $\hat{\lambda} = 1 - \tilde{\lambda}$. For tolerance levels close to $\frac{1}{2}$ the street turns into a black ghetto if blacks control more than 50 percent of the market as expected. But even if both groups can tolerate having up to 75 percent (80 percent) of their neighbors be of a different ethnicity, the share of blacks only has to exceed 70 percent (80 percent) for a black ghetto to arise. If blacks are strictly more tolerant than whites the critical values shift accordingly. A street with radius 2, for example, and tolerance levels of $\alpha_w = \frac{1}{2}$ and $\alpha_b = \frac{3}{4}$ will turn into a black ghetto as soon as the share of blacks exceeds 30 percent. Nevertheless, even though blacks are far more tolerant than whites, a black ghetto will dissolve again as soon as at least 80 percent of all apartment seekers are white.

Next, I look at the medium-run evolution of the residential neighborhood process in a setup where 5 percent of agents are completely tolerant, blacks constitute 80 percent of the housing market and the radius of interaction is again $r = 2$ or $r = 3$.[25] For the simulations in figure 8 I assume that both groups have equal tolerance levels which can only slow down the transformation of the area into a black ghetto, as compared to the case where whites are strictly less tolerant than

[24] An exception is the case where the tolerance levels of both ethnic groups are close to $\frac{1}{2}$. The heuristic argument of the previous section established that $\hat{\lambda} = \tilde{\lambda} = \frac{1}{2}$.

[25] The share $\lambda = 0.2$ has been chosen such that $\lambda < \hat{\lambda}$ in all cases.

Table 2: Comparison of waiting times $W(n, [0.9, 1])$ until more than 90 percent of residents are black starting from an all white neighborhood on a street $G_S^{r=2}(n)$ with $\lambda = 0.2$, $\alpha_b = \frac{3}{4}$, $\alpha_w = \frac{2}{3}$.

Expected Wait $W(n, [0.9, 1])$

	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.05$
n=100	9.32	7.16	5.84	4.86
n=1000	10.30	7.64	5.98	4.98
n=10000	10.02	7.84	6.00	5.00
n=20000	10.00	7.90	6.00	5.00

Estimated standard errors for the waiting times are 5% or less.

blacks.[26] The speed in which the all-white area is transformed into a black ghetto is impressive. It takes less than 5 tenant generations until more than 75 percent of the street has become black if blacks can tolerate having whites be two thirds or more of their neighbors. If blacks are less tolerant they will still populate about 50 percent of the area after 5 tenant generations.

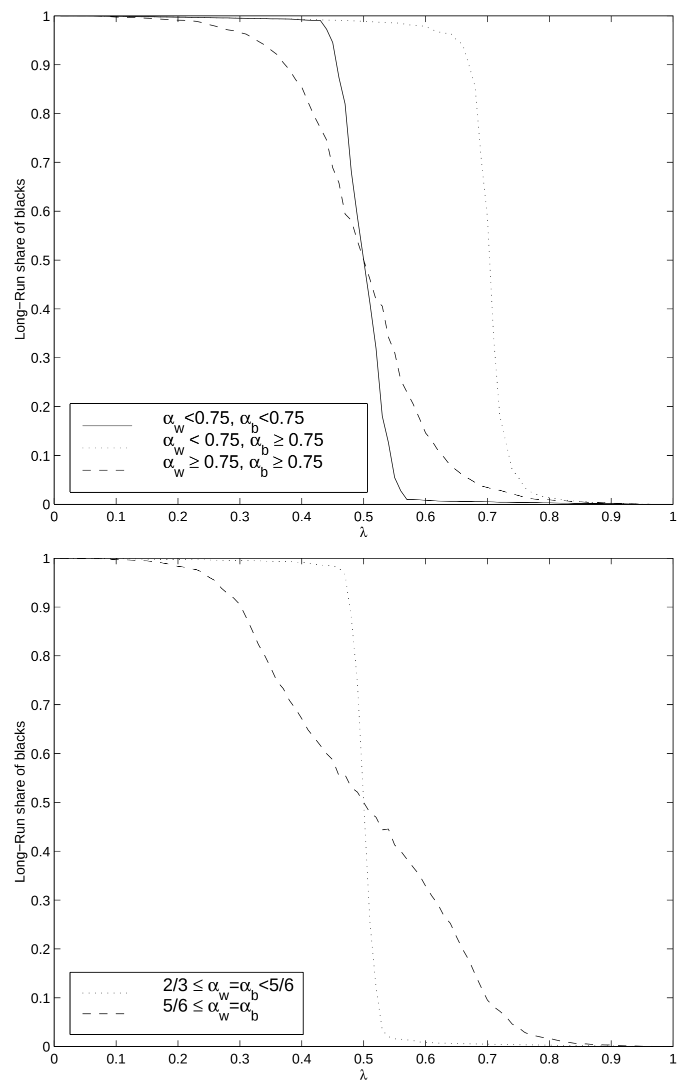
How does the speed of convergence depend on the share ϵ of tolerant agents and the size n of the street? Table 2 lists the expected waiting times $W(n, [0.9, 1])$ until 90 percent of all residents on the street are black for the geometry $G_S^{r=2}(n)$ and ϵ varying between 0.5 percent and 5 percent. The results indicate that the waiting time until convergence does not increase with the size of the street and depends only weakly on the share of tolerant agents.[27] Because of lemma 1, one would expect the waiting time not to depend on the size of street for large n. The simulations demonstrate, however, that the lemma holds even on fairly small streets.

Example: (cont. from section 2.3) By repeating the numerical exercise for streets one can directly compare the different behavior of the residential neighborhood process on bounded neighborhoods and on streets. I again assume that the black and white tolerance levels are $\alpha_b = \frac{3}{4}$ and $\alpha_w = \frac{2}{3}$ respectively, and that 5 percent of all agents are completely tolerant. The street is initially all-white when an influx of blacks into the market occurs ($\lambda = 0.2$). Table 3 reveals that convergence is now rapid even though the street is much larger than the bounded

[26] The less tolerant whites are, the easier it is for black clusters to expand on the street.

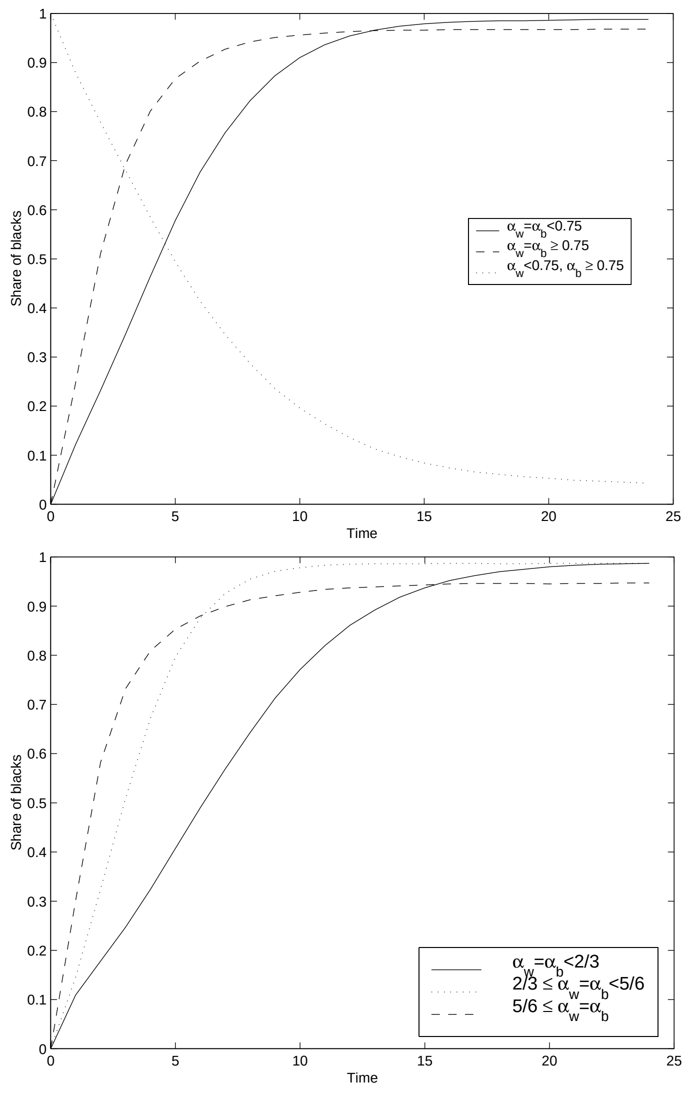
[27] Note that if $\epsilon = 0.005$ a vacant apartment with only white neighbors will switch color with a small probability of 2 percent. Still, it will only take 10 generations until 90 percent of all residents are black.

Figure 7: Dependence of long-run share of black residents on the share λ of whites in the housing market for streets with radius of interaction $r = 2$ (top) and $r = 3$ (bottom).



The size of the street has been set at $n = 8100$ and the share of tolerant agents at $\epsilon = 0.01$. The 'long-run share of blacks' was defined as the share of blacks at time $t = 10,000$. In all cases the street started off from a random configuration with 50 percent of residents being black.

Figure 8: Evolution of the share of black residents on a street $G_S^{r=2}(8100)$ (top) and $G_S^{r=3}(8100)$ (bottom). The process starts from an all-white neighborhood ($\lambda = 0.2$, $\epsilon = 0.05$) whenever blacks and whites are equally tolerant, and from an all black neighborhood ($\lambda = 0.8$, $\epsilon = 0.05$) if blacks are strictly more tolerant than whites.



Estimated standard errors are 5 percent or less.

Table 3: Comparison of waiting times $W(n, [0.97, 1])$ until 97 percent of all agents are black on various streets $G_S^r(5000)$ ($\alpha_b = \frac{3}{4}$, $\alpha_w = \frac{2}{3}$, $\epsilon = 0.05$, $\lambda = 0.2$)

Radius of interaction	r=2	r=4	r=6	r=8
Waiting time $W(n, I)$	6.94	7.16	7.68	7.56

Estimated standard errors for the waiting times are 5 percent or less.

neighborhoods I considered in the previous section. The example also illustrates why the resulting black ghetto lacks persistence on streets. If the balance in the housing market is reversed (i.e. 80 percent of all apartment seekers are white) the street $G_S^{r=2}(5000)$ will become 50 percent white within 5 tenant generations and 75 percent white within 10 generations, as the graph in figure 8 (top) illustrates.

3.2 Proof of Theorem 2

The proof utilizes some novel techniques from the theory of interacting particle system.[28] The argument proceeds in three steps. First, I propose a simplified Markov process σ_t on the street. Second, I show that the original process η_t and the simplified process σ_t can be *coupled* such that the hypothesis of theorem 2 only has to be proved for the simpler process σ_t. This last step is accomplished through lemma 3. Without loss of generality I restrict attention to the case when streets become black ghettos i.e. $\lambda < \frac{1}{2}$.[29]

The original neighborhood process is difficult to analyze because black clusters continuously form and break up when tolerant white tenants move into a vacant apartment. The simpler process σ_t limits and 'tags' all potential black clusters. Intuitively, the new process makes it both harder for new black clusters to form and easier for existing clusters to break up. The process is therefore biased against blacks in a monotonic fashion: if ghettos develop in the simplified process they should certainly develop in the original process, too.

Formally, the process σ_t is defined as follows. The street is divided up into k segments of fixed length N such that $n = kN$.[30] Residents move out at rate 1 and the process starts from an initial configuration σ_0 where all residents are white. The evolution of the process follows the same switching rules as before with the following qualifications:

1. I assume that a resident regards any neighbors outside his segment as white.

[28] Liggett (1985) provides a thorough introduction to this branch of probability theory.

[29] In this section it is no longer assumed that blacks are more tolerant than whites.

[30] I abstract away from integer constraints. I will take $n \to \infty$ and keep N fixed such that the contribution of a single segment of length less than N will vanish by the law of large numbers.

This implies that the dynamics of the process within each segment develops independently from other segments.

2. If a black cluster within the same segment already exists only the (at most two) adjacent white neighbors of the cluster can switch. This guarantees that no seeds for new disjointed clusters can be generated.

3. If a cell switches from black to white such that it divides a cluster up into two separate clusters, the smaller one dies.[31] Together with the previous rule, this assumption ensures that at any point in time at most one black cluster exists within each segment.

These rules completely define the evolution of the process starting from σ_0.

In general, coupling is simply a construction of two stochastic processes on a common probability space - in this case I construct a coupled process (σ_t, η_t) such that the two marginals of the process are the original process η_t and its simpler counterpart σ_t. In order to be of any interest the two processes cannot move independently but must be related in some nontrivial way. I define a simple partial order on the set Z of configurations of the street $G_S(n)$ which allows me to compare the two processes at any point in time:

$$\sigma \leq \eta \quad \text{if and only if} \quad \sigma(z) \leq \eta(z) \quad \text{for all residents } z \tag{6}$$

I assume that both processes start to evolve from the same initial all-white configuration $\eta_0 = \sigma_0$. The next lemma shows that there exists a coupled process (σ_t, η_t) such that the original process 'dominates' the new process monotonically, i.e. the inequality $\sigma_t \leq \eta_t$ holds with probability 1 for all $t \geq 0$.

Lemma 2 *There is a coupling (σ_t, η_t) such that both marginal processes start to evolve from the same all-white configuration $\eta_0 = \sigma_0$ and $\sigma_t \leq \eta_t$ holds with probability* 1 *at any point in time.*

Proof: see appendix D

One can immediately conclude that $E(f(\sigma_t)) \leq E(f(\eta_t))$ for each increasing function f on the space of configurations.[32] The share of blacks $X(\eta_t)$ of the residential neighborhood process at any time t, then first-order stochastically dominates the

[31] In the case of a tie I assume that the cluster clockwise to the right of the switching cell dies.

[32] Note that $E(f(\sigma_t)) = \int f(\sigma_t)\, d(\sigma_t, \eta_t)$ and $E(f(\eta_t)) = \int f(\eta_t)\, d(\sigma_t, \eta_t)$ because the coupled process has marginals η_t and σ_t. By construction $\sigma_t \leq \eta_t$ with probability 1 and therefore $f(\sigma_t) \leq f(\eta_t)$ with probability 1. This implies that $\int f(\sigma_t)\, d(\sigma_t, \eta_t) \leq \int f(\eta_t)\, d(\sigma_t, \eta_t)$.

share of blacks $X(\sigma_t)$ of the simplified process.[33] Therefore, the claim in theorem 2 only has to be established for the simplified process σ_t.

I exploit the observation that each segment of the street develops independently in the simplified process. Inside the initially all-white segment, a black cluster of length $[2r(1-\alpha_b)]^+$ will eventually form, which is the minimum length for the cluster to be stable under the undisturbed dynamics ($\epsilon = 0$).[34] Black house-seekers now show interest in the apartments surrounding this minimally stable cluster and the ends will start to move like a random walk with drift under the undisturbed dynamics. The drift is solely determined by the balance of the housing market.

I denote the expected long-run share of blacks in a segment of length N with $E_b(\epsilon)$. As $n \to \infty$ the number of segments becomes arbitrarily large, and by the law of large numbers and lemma 2 we can conclude for the long-run share $\tilde{X}_n$ of blacks in the original process that $P\left(\tilde{X}_n \in (2E_b(\epsilon) - 1, 1]\right) \to 1$. In order to finish the proof of theorem 2, the next lemma shows that the expected share of blacks in a segment can get arbitrarily close to 1 for sufficiently small ϵ and large N.

Lemma 3 *There is an upper bound $\hat{\lambda}(r, \alpha_w, \alpha_b) > 0$ such that for each $\lambda < \hat{\lambda}$ and each $\delta > 0$ there is an $\bar{\epsilon}$ such that for all $\epsilon < \bar{\epsilon}$ there is some N such that the expected share of blacks $E_b(\epsilon)$ in the segment of length N fulfills $E_b(\epsilon) > 1 - \delta$.*

Proof: see appendix E

The intuition for lemma 3 can be most easily outlined by using the language of stochastic stability analysis (see Kandori, Mailath and Rob (1993) and Young (1993)). If the share of blacks in the housing market is sufficiently large, the single black cluster living in a large segment of length N always exhibits a positive drift, i.e. is more likely to grow rather than to shrink. The process has essentially two 'limit sets' under the undisturbed dynamics: if the share of blacks is $x = 0$ the process can leave the basin of attraction only after a minimally stable cluster of size b has formed. On the other hand, if the share of blacks lies in a neighborhood I of $x = 1$ the process will escape that neighborhood only after a huge waiting time because of the positive drift pushing the process towards $x = 1$. Therefore both the shares $x = 0$ and $x \in I$ form 'limit sets' of the undisturbed dynamics. For any intermediate shares the dynamics will push the process rapidly into I, e.g. there are no further limit sets. It takes b 'mutations' until the basin of attraction of the limit set $x = 0$ can be left. The 'limit set' I can only be exited through a sequence of tolerant whites who move to vacant apartments inside the single black cluster.

[33]For a proof, define the following increasing function f_{x_0} indexed by each possible share of blacks: $f_{x_0}(\eta)$ is 0 if the share $X(\eta)$ of blacks in the configuration η is below x_0 and equals $X(\eta)$ otherwise.

[34]With $[x]^+$ I denote the smallest integer which is greater than or equal to x.

Each such 'mutation' cuts the length of the black cluster by at most half. Even after $b+1$ consecutive mutations its length will still be approximately $2^{-(b+1)}N$ and the undisturbed dynamics will push the process back into the 'limit set' I. It therefore takes fewer mutations to reach I than to leave I and we expect the process to spend most of its time inside I.

4 Rapid Segregation and Ghetto Persistence in Inner-City Areas

Ghettos can develop rapidly on streets in response to shifts in the housing market, as the previous section demonstrated. This mechanism overcomes the stasis in Schelling's (1972) original tipping model where the transition from an all-white steady state to a black ghetto does not occur within a realistic time frame even when the bounded neighborhood has only moderate size. However, the effect works in both directions, and periodic changes in the composition of the housing market give rise to "avenue waves". In order to reconcile the observed persistence of black ghettos in US cities with the dynamics of the model on streets one has to assume that African Americans have dominated the low-income housing market for the last 100 years. The data does not support this assertion because the migration of blacks to the cities has leveled off while other ethnic minorities (notably Mexican Americans) have grown at a far greater rate in recent decades. In the light of this evidence how can we explain the continued persistence of black ghettos even though blacks face far more competition in the housing market?

In an 'ideal' model the mechanism that gives rise to ghettos should be unidirectional, i.e. ghettos form rapidly but break up slowly. Inner-city areas can provide exactly such an environment if blacks are sufficiently more tolerant than whites. The following assumption on the tolerance levels of blacks and whites ensures that inner-cities preserve the ghetto formation mechanism of streets while making segregation persistent as on bounded neighborhoods.

Assumption 1 *Blacks can tolerate whites constituting 75 percent or more of their neighbors ($\alpha_b \geq \frac{3}{4}$), while whites can only tolerate blacks making up slightly more than 50 percent of their neighbors ($\alpha_w < \frac{1}{2} + \frac{r}{m}$).*[35]

The assumption will hold for the rest of this section.

Just as on streets, randomly forming black clusters can expand, quickly take over the inner-city area and therefore give rise to ghettos as long as blacks dominate the housing market sufficiently. If whites subsequently make up the majority

[35] The size of an individual neighborhood on the inner-city area $G_C^r(n)$ is m, i.e. $|N(z)| = m$. If the radius of interaction is $r = 1$ ($r = 2$) we have $m = 4$ ($m = 12$) and whites can tolerate at most two (seven) black neighbors.

in the housing market, however, random white clusters are hindered in their expansion. The two-dimensional geometry adds a 'geometric' drift that lets small white clusters shrink. This effect can be strong enough to completely counteract the pressure from the housing market which induces white clusters to expand on streets.

4.1 Rapid Formation of Black Ghettos in Inner-Cities

It can be easily checked that a black square cluster of size $(r+1) \times (r+1)$ can expand under the undisturbed dynamics, i.e. it can take over the inner-city area with positive probability even if there are no tolerant black house-seekers ($\epsilon = 0$).[36] This observation essentially guarantees that the inner-city will turn into a black ghetto, both in the medium and in the long run, if blacks dominate the housing market.

The formal proof of this claim exploits the results of the previous section by breaking the inner-city up into 'stripes' of width $r+1$, shown in figure 9. Although a stable $(r+1) \times (r+1)$ cluster can expand in all four directions in an inner-city, the ghetto formation mechanism will work even if it could only expand in East/West direction. Each stripe will then behave very much like a street: stable clusters of length $r+1$ form and take over the stripe within a waiting time of order $O(1)$ if blacks dominate the housing market. Moreover, the process will cluster around a black share of $x = 1$ on each segment and, hence, on the entire inner-city area. The proof of the next theorem goes through this reasoning in greater detail.

Theorem 3 *Given is an inner-city area* $G_C^r(n)$ *with group tolerance levels* α_w *and* α_b *satisfying assumption 1. Then there exists some critical value* $0 < \hat{\lambda}(r, \alpha_w, \alpha_b) < 1$ *such that the following holds when blacks dominate the housing market sufficiently* $(\lambda < \hat{\lambda})$.

1. *The inner-city becomes a black ghetto in the long run, i.e. the process clusters on the interval* $[x_b^*(\epsilon), 1]$ *with* $\lim_{\epsilon \to 0} x_b^*(\epsilon) = 1$.

2. *If the share* ϵ *of tolerant agents is sufficiently small the waiting time until the share of blacks exceeds* $1 - \delta$ *satisfies* $W(n, [1-\delta, 1]) = O(1)$.

Proof: The proof is easiest outlined for the case $r = 1$. The inner-city area is 'sliced' up into 'stripes' of length N and width $r+1 = 2$ (see figure 9). As in section 3.2 I construct a simplified process σ_t which evolves independently on each stripe. The switching rules are translated in a straightforward manner: they only differ in their emphasis on 'stripes' instead of 'segments'.

[36] Recall, that on streets the minimally stable black cluster had length $[2r(1-\alpha_b)]^+$.

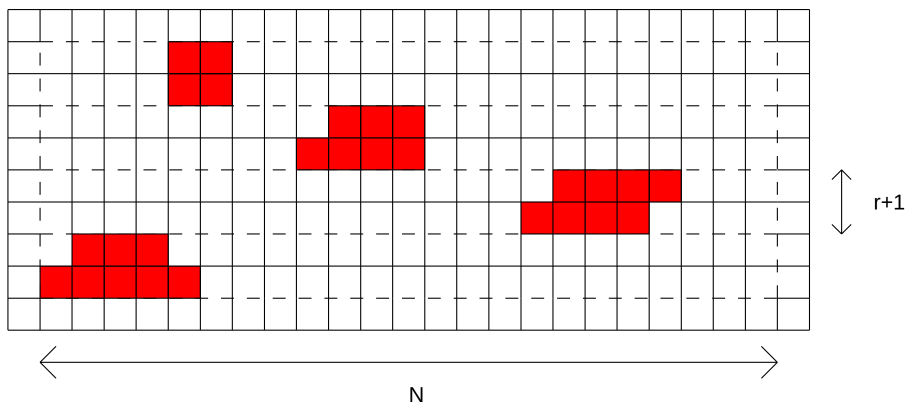
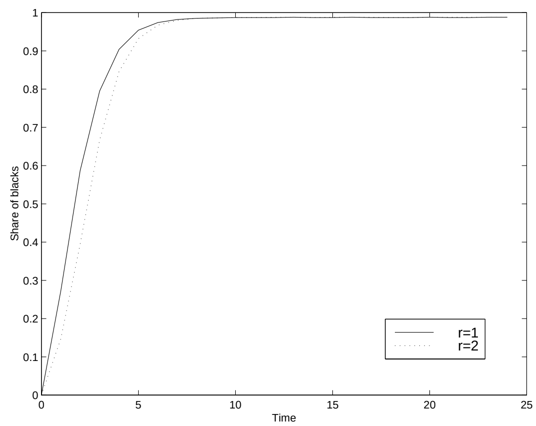


Figure 9: Four $N \times (r+1)$ stripes on an inner-city; the coupled process evolves independently within each stripe

1. A resident regards any neighbors outside her stripe as white. Therefore, the dynamics of the process within each stripe develops independently from other stripes.
2. If a black cluster within a stripe exists, only adjacent white neighbors can switch to black. A cluster has to 'fill up' vertically before it can expand horizontally. This rule implies that the single black cluster always fills up the full width of the stripe. The length of the single black cluster on a stripe can therefore be treated in the same way as the size of the single black cluster on a street segment.
3. If a cell switches from black to white inside the single black cluster the cluster is cut in two, and the shorter half is eliminated. This ensures that at any point in time there exists at most one black cluster within each segment.

Due to assumption 1, a black cluster can only be invaded by whites under the undisturbed dynamics at its two boundaries. Any apartment inside a black cluster with a distance of at least r from either boundary has a black neighborhood share of $\frac{1}{2} + \frac{r}{m}$ which exceeds the white tolerance level α_w. Apartments at the boundary of the cluster, on the other hand, have a black neighborhood share of at least 25 percent, which makes them acceptable to all black house-seekers. Each cluster behaves like a cluster on a street segment: it can expand under the undisturbed dynamics with a drift depending on the composition of the housing market, and it can only be broken in half by rare ϵ-jumps. The proofs of theorem 2 and lemma 1 can be easily adapted to establish the existence of $\hat{\lambda} > 0$ and fast convergence. QED

Figure 10: Evolution of the share of black residents in the inner-city areas $G_C^{r=1}$ (8100) and $G_C^{r=2}$ (8100). The process starts from an all white neighborhood with tolerance levels $\alpha_b = \frac{3}{4}$ and $\alpha_w = \frac{7}{12}$ ($\lambda = 0.2$, $\epsilon = 0.05$).



Estimated standard errors are 5 percent or less.

Monte-Carlo simulations confirm that black ghettos form as rapidly in inner-cities as they do on streets. Figure 10 illustrates the medium-run evolution of the residential neighborhood process in inner-cities with radius of interaction $r = 1$ and $r = 2$ where, again, 5 percent of agents are tolerant and blacks constitute 80 percent of the housing market. In both cases I have set the white and black tolerance levels at $\alpha_w = \frac{7}{12}$ and $\alpha_b = \frac{3}{4}$ such that they just satisfy assumption 1. Within five tenant generations 90 percent of all residents are black, which is almost the same waiting time I obtained for streets (see figure 8).

4.2 "Encircling" and Persistence of Ghettos in Inner-Cities

On streets the mechanism that gives rise to ghettos is fully reversible. In response to an influx of white apartment-seekers, white clusters can form and rapidly break up the black ghetto. The fact that blacks have a far higher tolerance level than whites only matters insofar as whites have to dominate the housing market relatively more in order to break up the all-black equilibrium ($\tilde{\lambda} > 1 - \hat{\lambda}$). However, the residential neighborhood process behaves in a qualitatively different way in inner-cities. 'Small' white clusters can nolonger expand under the undisturbed dynamics.

I consider a white cluster inside a black inner-city ghetto to be 'small' if it is "encircled" by black residents, i.e. it does not span the residential area.[37] Such

[37]Formally, I call a cluster of residents "encircled" in an inner-city with radius of interaction r

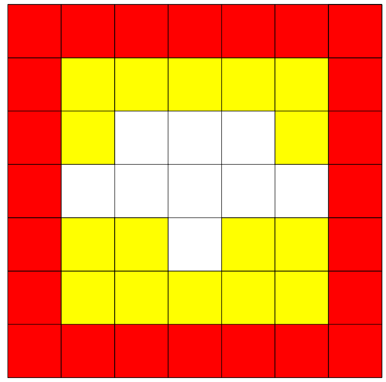


Figure 11: Isolated white cluster in an inner-city area $G_C^{r=1}$ (49). The cluster can be covered by a 5×5 rectangle (lightly shaded).

a cluster can be covered by a rectangle which is convex in the two-dimensional geometry. For this reason, each apartment vacated by a black resident along the boundary of this rectangle has more black than white neighbors. More precisely, a vacant apartment outside the rectangle has a black neighborhood share of at least $\frac{1}{2} + \frac{r}{m}$ which exceeds the tolerance level of whites. Close to the corners of the rectangle the black share even approaches 75 percent. Therefore, the white cluster can never expand beyond the rectangle unless tolerant house-seekers move in along the boundaries. Black house-seekers, on the other hand, can easily invade the white cluster under the undisturbed dynamics ($\epsilon = 0$). Obviously, the cluster has to die out in this environment and can never take over the inner-city regardless of the balance in the housing market as the following lemma shows.

Lemma 4 *Under assumption 1 and without the presence of tolerant agents* ($\epsilon = 0$) *an "encircled" white cluster in an inner-city area* $G_C^r(n)$ *will die out almost surely for any balance in the housing market* $0 < \lambda < 1$.

Proof: see appendix G

A white cluster can therefore only survive under the undisturbed dynamics and lie outside the basin of attraction[38] of the black ghetto configuration if it is 'large' and nolonger encircled. The stable 'large' cluster shown in figure 12 serves as an example.

if the cluster can be covered by a rectangle with width and length not exceeding $\sqrt{n} - 1 - r$. This ensures that the dynamics of the process along the boundary is not influenced by the finiteness of the inner-city.

[38]The basin of attraction $D(\Omega)$ of some subset of configurations $\Omega \subset Z$ is the set of configurations from which the undisturbed process reaches an element of Ω with probability 1.

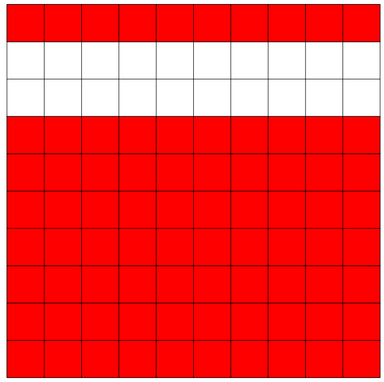


Figure 12: 'Large' non-encircled white cluster in an inner-city with radius of interaction $r = 1$

The minimum number of completely tolerant house-seekers who have to move into the black ghetto in order to form a non-encircled white cluster grows with $\sqrt{n}$. On streets, on the other hand, the minimally stable white cluster has only size $[2r(1-\alpha_w)]^+$, which does not depend on the size of the street. Intuitively, it should therefore take longer to leave a black inner-city ghetto than leave a black street ghetto if the share ϵ of tolerant agents is small. The next theorem confirms this insight by comparing the waiting times until the share of white residents exceeds some share $\delta < 1$ in an inner-city and on a street *of the same size* n.

Theorem 4 *Given are an inner-city area $G_C^r(n)$ and a street $G_S^r(n)$ of equal size n. Assumption 1 on the black and white tolerance levels holds, and the share ϵ of tolerant agents is small. Then the waiting time until the share of whites exceeds δ satisfies $W_C(n,[0,1-\delta]) \geq \epsilon^{-\left[\frac{\sqrt{n}}{r+1}\right]}$ in the inner-city area and $W_S(n,[0,1-\delta]) \sim \epsilon^{-[2r(1-\alpha_w)]^+}$ on the street.*

Proof: see appendix H

The ratio $\frac{W_C(n,[0,1-\delta])}{W_S(n,[0,1-\delta])}$ of the waiting times to leave an inner-city and street ghetto respectively, can become arbitrarily large if there are few tolerant house-seekers and the size of the residential area is large. The "encircling phenomenon" therefore lends persistence to black inner-city ghettos, and makes the ghetto formation process uni-directional. Inner-cities are in some sense a 'hybrid' geometry because they combine features of streets and bounded neighborhoods.[39]

[39] A comparison of the waiting times to leave a black bounded neighborhood ghetto and an inner-city ghetto of equal size n also reveal that inner-cities lie somehow 'between' streets and bounded neighborhoods. It takes $[(n-1)(1-\alpha_w)]^+$ 'mutations' to leave the basin of attrac-

Table 4: Comparison of waiting times $W(n, [0, 0.5])$ until more than 50 percent of residents are white starting from a black ghetto on a street and an inner-city area with individual neighborhoods of equal size m ($\lambda = 0.8$, $\alpha_b = \frac{3}{4}$, $\alpha_w = \frac{7}{12}$, $n = 8100$).

Expected Wait ($m = 4$)

	$\epsilon = 0.005$	0.01	0.015	0.02	0.03	0.04	0.05
$G_S^{r=2}(n)$	57.4	26.9	17.3	12.8	8.5	6.8	5.4
$G_C^{r=1}(n)$	594.5	63.7	27.3	16.9	9.8	7.1	5.8

Expected Wait ($m = 12$)

	$\epsilon = 0.005$	0.01	0.015	0.02	0.03	0.04	0.05
$G_S^{r=6}(n)$	1618	267.4	94.0	48.5	19.9	11.4	8.1
$G_C^{r=2}(n)$	$> 3 \times 10^4$	2912.3	169.0	66.4	21.1	11.8	8.0

Estimated standard errors for the waiting times are 5% or less.

Theorem 4 does not allow us to assess if the increase in the persistence of ghettos in inner-cities compared to streets is quantitatively significant. In particular, the share ϵ of tolerant agents should not be unrealistically small for the effect to apply. For this purpose I have relied on simulations in order to calculate the waiting times $W(0, [0, 0.5])$ until at least 50 percent of all residents are white. Table 4 compares the waiting times for the street/ inner-city pairs $(G_S^{r=2}(n), G_C^{r=1}(n))$ and $(G_S^{r=6}(n), G_C^{r=2}(n))$, respectively. These pairs have been chosen so that individual neighborhoods have equal size in both geometries, i.e. $m = 4$ and $m = 12$. In both cases I have again set the white and black tolerance levels at $\alpha_w = \frac{7}{12}$ and $\alpha_b = \frac{3}{4}$. I have assumed that 80 percent of all house seekers are white, and I have varied the share of tolerant agents between one half of a percent and 5 percent. This implies that a vacant apartment with only black neighbors will switch color with a probability between 2 percent and 17 percent which I consider reasonably large. The waiting times for the inner-cities are consistently larger than the corresponding waiting times on streets. For $\epsilon = 0.02$ the inner-cities are more than 30 percent more persistent while for $\epsilon = 0.01$ the difference is at least three-fold.

tion of a black ghetto in a bounded neighborhood. As in theorem 4 one can then show that $\frac{W_B(n,[0,1-\delta])}{W_C(n,[0,1-\delta])} \to \infty$ as $\epsilon \to 0$ because n grows faster than $\sqrt{n}$. Inner-cities are therefore more persistent than streets but less persistent than bounded neighborhoods.

5 Historical Evidence

This section evaluates the empirical relevance of my model for explaining segregation in the US. The main prediction of the theory is the rapid formation of ghettos on streets and in inner-cities as a response to shifts in the housing market. As a natural first-pass test I compare the degree of residential segregation over the last 200 years with changes in the relative demand for housing between African Americans and non-blacks. This exercise provides broad support for the model. In order to test the more subtle implications of the theory I take a closer look at the rise of the black ghetto in Harlem. In particular, I find that the dynamics of Harlem's transformation conform with the contagious growth process predicted by the theory, which has random black clusters form and then expand along the boundaries. Although I concentrate on the residential separation between African Americans and non-blacks for most of this section I provide evidence for "avenue waves" in Chicago which involved both African Americans and European immigrants.

5.1 The Rise of the Black Ghetto at the Turn of the Century

After the Civil War race relations improved, and northern cities such as Cleveland, Philadelphia and Chicago established integrated schools, hospitals and colleges.[40] Throughout the 19th century segregation between African Americans and non-blacks was relatively low, and the average African American lived in a ward that was only 20 percent black.[41] As far as spatial concentrations of blacks existed, they had not stabilized yet: in New York, for example, the principal clusters of black concentration moved repeatedly over the century.

Between 1860 and 1890 the share of blacks in fact decreased in many of the booming northern cities, and blacks made up only 2.5 percent of the population in the North and Midwest in 1890. Blacks started to leave the South and migrate to the booming North in significant numbers only after 1890.[42] Even then, the black growth rate just about matched the rate of increase of the general population, such that the share of blacks in the housing market was likely to be low. This combination of a low degree of segregation, shifting clusters of black concentration and an insignificant presence in the housing market before the turn of the century is consistent with my model when the share of blacks is non-critical, i.e. smaller

[40] Cleveland, for example, integrated schools in 1871 (Kusmer 1976).

[41] See table 2 in Cutler, Glaeser and Vigdor (1997).

[42] It is puzzling that few blacks left the South between 1865 and 1890 even though economic conditions were poor. Kusmer (1976) argues that the first black generation born in freedom had a different perspective from their parents and sought to exploit improved economic opportunities more actively.

than the share $1 - \hat{\lambda}$ which has to be exceeded to give rise to ghettos.

This picture changed radically with the outbreak of the first World War in 1914. Immigration to the US fell off sharply and never recovered after the war due to the immigration restriction enacted in the 1920s. The manufacturing industry continued to demand cheap labor, and companies began to dispatch labor agents to the South in order to convince more African Americans to move north. These factors resulted in a massive population movement between 1916 and 1919 known as the *Great Migration*[43] that continued well into the second half of the 20th century. The annual growth rate of the black urban population in the North was 3.1 percent between 1910 and 1940 and 4.4 percent between 1940 and 1970.[44]

Competition for housing intensified because the black community grew at a much faster rate than the general population in northern cities. In Chicago, for example, 18.5 percent of the net inflow of newcomers between 1920 and 1930 were African American, and in Cleveland the corresponding share was an impressive 36.2 percent.[45] As blacks were poorer than the average newcomer, they presumably dominated the market for apartments at the lower end of the socio-economic scale far more than these numbers indicate. It is likely that the share of blacks did not even have to exceed 50 percent to transform a residential area into a black ghetto[46] because evidence from surveys[47] shows consistently that blacks have a higher tolerance level than whites. Therefore, it is perfectly plausible that the share of blacks in northern cities was high enough during this period to trigger the rapid formation of ghettos in inner-cities and on streets as predicted by the theory.

In the wake of the Great Migration, northern US cities, in particular, became indeed much more segregated. In 1940 the average African American lived in a residential area that was 37.6 percent black and by 1970 that share had increased to almost 70 percent. Cutler, Glaeser and Vigdor (1997) found in a sample of 313 US cities that only 5 cities had ghettos in 1910 but more than a third had one by 1970.[48] Most of these almost exclusively black neighborhoods formed around the principal black cluster of concentration that happened to exist before the Great Migration.

In my model decentralized racism is the sole transmission channel that translates the conditions in the housing market during and after the first World War

[43] In these years alone the number of African Americans doubled in Cleveland, tripled in Chicago and increased more than sixfold in Detroit (Kusmer 1976).

[44] See table 2 in Cutler, Glaeser and Vigdor (1997).

[45] See table 1 in Spear (1967) for Chicago and table 1 in Kusmer (1976) for Cleveland.

[46] In section 3.1 I found that a street $G_S^{r=2}$ with black and white tolerance levels $\alpha_b = \frac{3}{4}$ and $\alpha_w = \frac{1}{2}$ becomes a ghetto as soon as the share of blacks exceeds 30 percent.

[47] Cutler, Glaeser and Vigdor (1997) cite evidence from the General Society Survey.

[48] The authors characterize a city as having a ghetto if the index of dissimilarity is greater than 0.6 and the index of isolation exceeds 0.3 (see the paper for details on calculating these standard measures).

into changes in the level of segregation. Although other factors undoubtedly contributed to this process, I argue that decentralized racism was the most significant channel during the early formative years. First, sorting by socio-economic differences explains less than half of the observed variation in segregation indices between neighborhoods, even in the 1950s (Taeuber and Taeuber 1965). Second, collective-action racism[49] could not prevent the invasion of white neighborhoods at the onset of the Great Migration, as the example of Harlem shows. African Americans only made up a small share of the population, and the relatively peaceful race relations were only gradually overshadowed by resentment due to continued black migration from the South. Even neighborhoods that remained largely white during this period experienced scattered instances of black families moving in and out according to an unpredictable pattern.[50] Such noisy "reshuffling" of blacks in 'white ghettos' does not fit a theory of collective racism but is consistent with my model in the case of up-scale neighborhoods, for example, which few blacks could afford and where whites would face little competition in the housing market.

There is stronger evidence that collective-action racism played a more significant role by the middle of the century in sustaining segregation. By then, ghettos were already well-established, and a formal and informal institutional framework consolidating segregation had developed. Cutler, Glaeser and Vigdor (Cutler, Glaeser, and Vigdor 1997) find that in 1940, blacks paid relatively more for equivalent housing than whites in more segregated cities, as compared to less segregated cities. While this observation is consistent with some degree of collective-action racism, it disappeared from the data by 1990. Nowadays whites pay more for equivalent housing than blacks in more segregated areas, suggesting that decentralized racism is again the driving force behind continuing segregation.

Segregation has slightly declined since the 1970s because formerly all-white neighborhoods have become more racially mixed. But almost exclusively black ghettos persist and show very little sign of change. The stability of ghettos is surprising because blacks no longer dominate the housing market as they did in the aftermath of the Great Migration. After 1970 the black community in the northern cities only increased at an annual rate of 0.9 percent. While the share of blacks has stagnated other ethnic groups have shown vigorous growth. In particular, the share of Hispanics in US cities doubled between 1970 and 1990 to 10.3 percent.[51]

[49] An important type of collective-action racism were racial zoning or restrictive covenants that excluded blacks from particular residential areas (Massey and Denton (1993)).

[50] Smith (1959) compared the census data for New Haven between 1940 and 1950 and found that 76 blocks with a black share of less than 10 percent became all-white, while black families moved into 72 formerly all-white blocks. One third of these new blocks were contiguous to those which they were replacing and the rest were scattered throughout the city and lacked any spatial pattern.

[51] Mexican Americans are also highly segregated and live in neighborhoods with a Mexican share of 50.3 percent (Borjas (1995), table 4).

Inner-cities can provide an explanation for the longevity of black ghettos despite the fact that African Americans face far more competition in the housing market. The theory also suggests that today's inner-city ghettos are unlikely to disappear in the near and even medium term unless they are forcibly broken up by some policy intervention, such as urban redevelopment.

5.2 Harlem's Transformation into a Black Ghetto

Harlem was an affluent suburb of New York City in the 19th century and became the largest black ghetto in the US by 1920. The various stages in the spatial growth of the ghetto are well documented, which allows me to directly test the dynamic predictions of my theory on streets and in inner-cities, i.e. the growth of randomly forming black cluster around their boundary.[52]

New York City provides an almost ideal environment for the application of my model. The residential turnover rate was high because former peripheral neighborhoods, such as Harlem, were continually redeveloped as the metropolis expanded northwards on Manhattan island. Harlem, for example, was a rural village and became incorporated only in 1873. By 1886, the three lines of the elevated railroad came as far north as 129th Street, and a massive building boom in the 1880s made Harlem a preferred residential area for New York City's white upper- and upper-middle-classes. Lower Harlem experienced an influx of Eastern European Jews in the 1890s and of Italians before 1890 (see the map of Harlem in figure 13).

Like other northern cities, New York was not particularly segregated in the 19th century. African Americans did not dominate any single neighborhood, and the principal clusters of black concentration moved repeatedly up the West Side over the course of the century.[53] In 1890, for example, six wards in Manhattan had a substantial black population of between 2,000 and 4,000.

There were few scattered black families in Harlem before the turn of the century. Most of them were servants and lived at the periphery of white Harlem. Blocks occupied by African Americans in 1902 are marked in figure 13. African Americans entered Harlem in greater numbers during the years 1900-1914, when the black population of Manhattan doubled. Development in West Harlem north of 130th Street had been slow in the 1890s because the area lacked public transportation. The construction of the Lenox subway line up to 145th Street in the years 1898 to 1904 set off a building boom and massive speculation in property along Lenox

[52] This section is based on Osofsky's (1963) comprehensive history of Harlem (especially chapters 5-8).

[53] In the early 19th century many blacks lived in the Five Points district on the site of the present City Hall. By 1860 the district was overwhelmingly Irish and the largest cluster of African Americans could be found in Greenwich Village. Between 1880 and 1890 their numbers declined as the district became predominantly Italian. San Juan Hill and the "Tenderloin" (between 20th Street and 53rd Street) emerged as the most populous black residential areas.

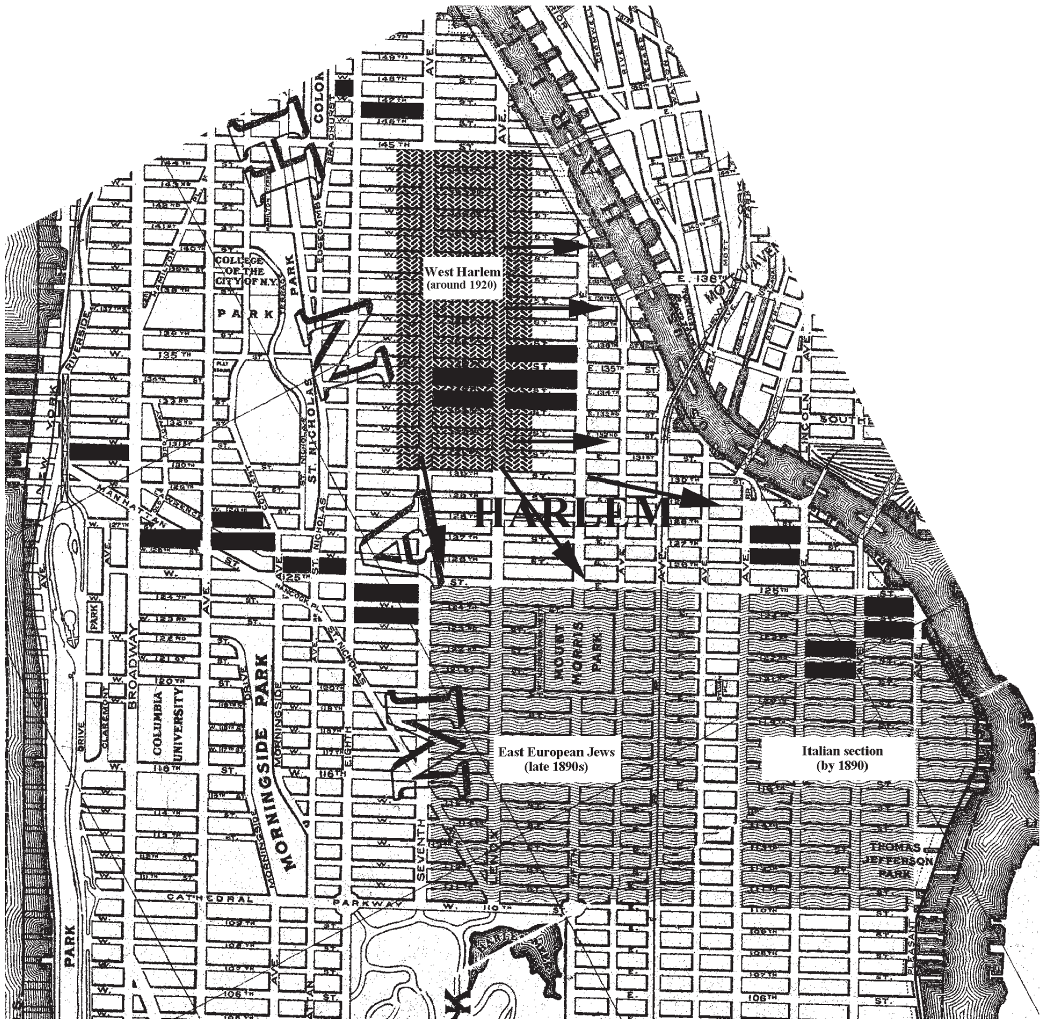


Figure 13: Harlem was an essentially all-white middle and upper-class neighborhood in the 19th century. Lower Harlem saw an influx of Eastern European Jews in the 1890s and had an Italian section in the south-east. Only a few scattered black families lived in blocks at the periphery of central Harlem which are marked black in the figure. West Harlem became a black ghetto until 1920 (darkly shaded). The ghetto expanded in the 1920s to the east and south as indicated by the arrows. Blacks lived as far south as 110th Street by the end of the decade.

and Seventh Avenue where entire new apartment blocks were built. By 1904-5 the bubble burst and realtors woke up to the fact that there was insufficient demand for these high-quality apartments. Landlords began to compete intensely for tenants and some of them started to open their apartment houses to blacks. Demand for decent housing was strong amongst African Americans during this period. More and more blacks migrated to New York City, and established blacks were displaced from their old living quarters in Lower Manhattan as the business district expanded.[54]

This combination of factors colluded to establish the initial black cluster of residents in Harlem which then expanded dramatically in the aftermath of the Great Migration. Osofsky (1963, page 17) emphasizes that Harlem's black colony would most likely have been a passing phenomenon just like previous clusters of black concentration in Lower Manhattan without the enormous influx of black migrants after 1900:

> The most important factor underlying the establishment of Harlem as a Negro community was the substantial increase of Negro population in New York City in the years 1890-1914. That Harlem became the specific center of Negro settlement was the result of circumstance; that *some* section of the city was destined to become a Negro ghetto was the inevitable consequence of the Negro's migration from the South.

African Americans took over West Harlem in a striking geographical pattern which resembles contagious growth. There was a clear 'color line' that separated the southward advancing black settlement from established white residents. This type of dynamics is not only predicted by my theory but, more generally, suggests that decentralized local interaction between residents should be at the heart of any realistic model of segregation. Landlords attempted to stop the black invasion through collective-action racism and successively signed restrictive agreements on West 140th, 137th, 135th, 131st, 129th Streets etc., which obliged them not to rent to blacks. Each of these local arrangements ultimately failed because sooner or later some landlord would 'panic-sell' and the coalition collapsed. Restrictive agreements also invited 'block-blusters' to test the strength of support for unified action. These speculators bought single apartment houses on an all-white street and invited black tenants. Adjoining white owners then had to re-purchase the apartment house at inflated prices in order to evict the black tenants again.

By 1920, an almost exclusively black ghetto had formed north of 130th Street and West of 5th Avenue as shown in figure 13. The ghetto expanded further to the

[54] Many black apartment blocks disappeared when Pennsylvania Station was built in the Tenderloin at the beginning of the century. In 1914 African Americans occupied about 1,100 different houses within a 23 block area of Harlem and 80% of the whole black population lived in Harlem by then.

east and south and completely crowded out the white residents in central Harlem. By the end of the 1920s African Americans lived as far south as 110th Street and the ghetto consolidated in the subsequent decades as black migration continued.

5.3 "Avenue waves" in Chicago

I have emphasized that ghettos are not persistent on streets because the ghetto formation process is reversible. The model therefore predicts "avenue waves" in response to periodic shifts in the housing market. In the 19th century there existed a natural source of variation because different groups of immigrants entered the country at different points in time. Immigration in the first half of the century was characterized by waves from Northern Europe while in the latter half of the century immigrants from Southern and Eastern Europe dominated. If one assumes that most immigrants arrived poor in the US and climbed the social hierarchy at similar rates one would expect that different ethnic waves of immigrants joined the housing market for residential areas of a particular quality at different points in time. This, in turn, induced shifts in the balance λ of the housing market.[55]

Burgess (1928) recorded the resulting avenue waves in Chicago as shown in figure 14. He notes that "the great arterial business streets of the city have been and remain the highways of invasion". Of particular interest are the A and B waves where 'new' immigrant groups (Hungarians, Italians and Poles) crowd out 'older' immigrants (Germans and Scandinavians).

6 The Relationship between Stochastic Stability and Clustering

My analysis characterized the evolution of the residential neighborhood process through clustering rather than the standard stochastic stability techniques developed by Kandori, Mailath and Rob (1993) and Young (1993). This section explores the relationship between both techniques and concludes that stochastic stability can seriously mispredict the long-run behavior of a stochastic system because it ignores too much information about its undisturbed dynamics. The problem is most severe for the large-scale systems which we typically encounter in evolutionary environments. In the context of my model I show that stochastic stability fails to predict the rise of ghettos on streets because it does not take into account the balance in the housing market.

[55] The "avenue waves" effect should be strongest for streets at the lower socio-economic end of the market. The diffusion rate at which residents move into better neighborhoods differs by ethnicity and between individuals such that ethnic waves should become increasingly intermingled.

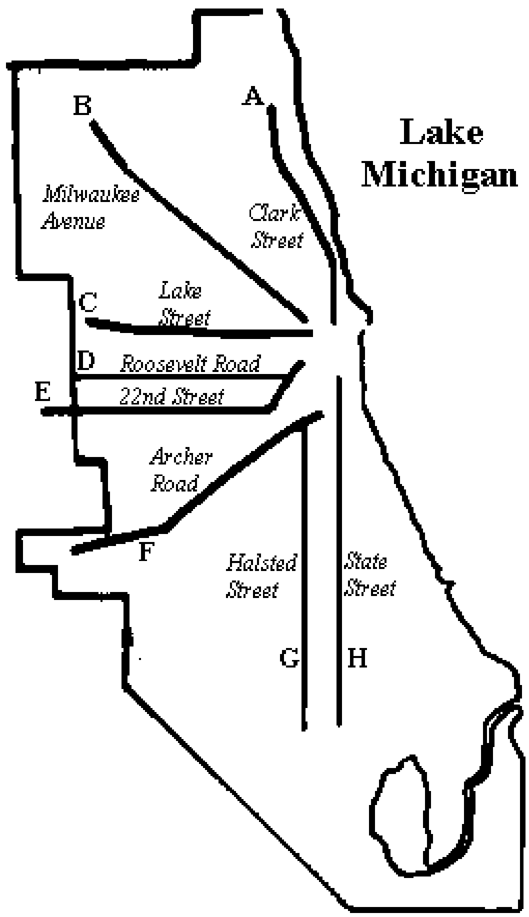


Figure 14: Chicago's avenue waves (Burgess, 1928) - Germans/Scandinavians followed by Hungarians/ Italians (A), Germans/ Scandinavians followed by Poles (B), invasion of African Americans (C), invasion of Russian Jews (D), invasion of Czechs (E), Polish invasion (F), invasion of Irish (G), Invasion of African Americans (H)

But it is my hope that clustering will enhance our understanding of both future and existing evolutionary models. I present two examples which are intended to illustrate the robustness of clustering versus stochastic stability. First, I consider a simple model of imitation where agents live on a general (not necessarily regular) class of graphs and hold one of two possible 'opinions'. Whereas stochastic stability predicts that agents synchronize their opinions in the long run, clustering analysis reveals that, to the contrary, a system of many agents will typically be in disagreement. Second, I revisit a well-known large population coordination game which has been studied by Ellison (1993). Although both clustering and stochastic stability predict the long-run behavior of the large-scale system correctly, in this case the standard technique is vulnerable to seemingly innocuous changes in the dynamics.

I will rely on a characterization of stochastic stability recently introduced by Ellison (1999). His waiting-time approach, unlike the "tree-surgery" arguments used in the earlier papers, not only makes the reasoning behind stochastic stability arguments very transparent but also allows me to clearly identify the key weakness of this concept. Ellison considers a general 'model of evolution' (Z, P, P^{ϵ}) with a state space Z and a Markov process defined over Z in discrete time[56] with 'disturbed' transition matrix P^{ϵ} and 'undisturbed' transition matrix P. The matrix is assumed to be ergodic for each $\epsilon > 0$ and, P^{ϵ} is continuous in ϵ such that $P^0 = P$. Ellison then defines a cost function $c : Z \times Z \to R^+ \cup \infty$ such that for all pairs of states $\eta, \eta' \in Z$, $\lim_{\epsilon \to 0} P^{\epsilon}_{\eta\eta'} / \epsilon^{c(\eta,\eta')}$ exists and is strictly positive if $c(\eta, \eta') < \infty$ (with $P^{\epsilon}_{\eta\eta'} = 0$ for sufficiently small ϵ if $c(\eta, \eta') = \infty$). Intuitively, the cost of transition can be thought of as the number of independent mutations necessary for it to occur.

Ellison introduces two new concepts, the radius and the coradius, which he uses to bound the waiting times required to leave and enter the basin of attraction of a union of limit sets[57] $\Omega \subset Z$. The radius $R(\Omega)$ describes the minimum cost of leaving the basin of attraction $D(\Omega)$ and is a measure of the persistence of the process when it rests at Ω. Formally, Ellison defines a path out of $D(\Omega)$ as a sequence of distinct states $(\eta_1, \eta_2, .., \eta_T)$ with $\eta_1 \in \Omega$, $\eta_t \in D(\Omega)$ for $1 < t < T$ and $\eta_T \notin D(\Omega)$. The set of all these paths is denoted $S(\Omega, Z - D(\Omega))$. The radius can then be defined as

$$R(\Omega) = \min_{(\eta_1,..,\eta_T) \in S(\Omega, Z - D(\Omega))} \sum_{t=1}^{T-1} c(\eta_t, \eta_{t+1}). \tag{7}$$

[56] The residential neighborhood process is defined in continuous time. Appendix A illustrates how such a process can be transformed into a corresponding discrete time process such that all of Ellison's results carry over.

[57] The limit sets or recurrent classes of a stochastic system are the sets of states which can persist in the long run absent noise or mutations ($\epsilon = 0$).

The coradius $CR(\Omega)$, on the other hand, captures the length of time necessary to reach the basin of attraction of Ω starting from any other state by counting the number of intermediate mutations:[58]

$$CR(\Omega) = \max_{\eta_1 \notin \Omega} \min_{(\eta_1, \ldots \eta_T) \in S(\eta_1, \Omega)} \sum_{t=1}^{T-1} c(\eta_t, \eta_{t+1}) \tag{8}$$

A combination of a large radius $R(\Omega)$ and a small coradius $CR(\Omega)$ ensures that the process reaches the basin of attraction $D(\Omega)$ quickly, but is very reluctant to leave it. Building on this intuition Ellison can prove the following theorem:

Theorem 5 *The union of limit sets Ω is stochastically stable if $R(\Omega) > CR(\Omega)$. The waiting time to leave the basin of attraction $D(\Omega)$ is $W(\Omega, Z - D(\Omega), \epsilon) \sim \epsilon^{-R(\Omega)}$, and the process reaches a long-run equilibrium after a waiting time of $W(\eta, \Omega, \epsilon) = O\left(\epsilon^{-CR(\Omega)}\right)$ for any $\eta \notin \Omega$.*

Proof: see Ellison (1999)

The waiting time approach has the advantage of giving a bound on the rate of convergence to the long-run equilibrium. In models of local interaction the coradius typically remains small even if the system is large (as in Ellison (1993)). This observation is then interpreted as evidence that local interaction 'speeds up' convergence to the long-run equilibrium.

Unfortunately, the coradius can give a very misleading picture of how fast the process actually reaches the basin of attraction $D(\Omega)$. The random walk example discussed in the introduction illustrates the problem nicely (see figure 1). It is easily shown that only the state $n+2$ is stochastically stable because it takes at most one mutation to reach its basin of attraction, but two mutations to leave it (i.e. $R(\{n+2\}) = 2$ and $CR(\{n+2\}) = 1$). Theorem 5 also indicates that the process reaches its long-run equilibrium quickly because the waiting time is of the order $O\left(\frac{1}{\epsilon}\right)$ independent of the 'size' n.

The example is simple enough to carry out a more careful analysis. Although the process will spend almost all its time at $n+2$ as $\epsilon \to 0$, it is instructive to calculate how small ϵ has to be depending on the size n of the system in order to find the process in a small δ-neighborhood $[(1-\delta)n, n+2]$ of the long-run equilibrium with probability $\gamma > 0$. The following condition on ϵ and n has to be

[58] Ellison also defines the modified coradius $CR^*(\Omega)$ which bounds the waiting time until convergence more precisely. For the purpose of comparing clustering and stochastic stability, however, is suffices to use the simple coradius.

satisfied:[59]

$$\frac{2\frac{\epsilon}{1-\epsilon}\left(\left(\frac{1}{2}\right)^{(1-\delta)n}-\left(\frac{1}{2}\right)^{n}\right)+\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{2}\right)^{n-1}\frac{1-\epsilon}{\epsilon}}{1+2\frac{\epsilon}{1-\epsilon}\left(1-\left(\frac{1}{2}\right)^{n}\right)+\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{2}\right)^{n-1}\frac{1-\epsilon}{\epsilon}}\geq\gamma$$

For large n this condition becomes approximately

$$\left(\frac{1}{2}\right)^{n-1}\frac{1-\epsilon}{\epsilon}\geq\frac{\gamma}{1-\gamma}, \tag{9}$$

which requires that $\epsilon<\overline{\epsilon}_n=\left(\frac{3}{2}\right)^{-n}$.[60]

Therefore, stochastic stability describes the long-run behavior of the random walk well on large systems only if the noise term ϵ is extremely small. Using the technique of appendix B it is straightforward to show that the waiting time until convergence is at least of the order 3^n even though it takes just a single mutation to reach the δ-neighborhood. Convergence to the stochastically stable equilibrium is therefore anything but fast.[61]

What has gone wrong? Stochastic stability analysis essentially ignores the nature of the undisturbed dynamics. In my simple example it suffices that there is *some*, however small, positive probability of reaching state n starting from state 1. This transition has zero cost attached to it and hence does not enter the calculation of the coradius. But the larger the size n of the system, the more the undisturbed dynamics pushes the process away from state n, and the error term ϵ has to decrease at an exponential rate in order to sustain the predictions of stochastic stability. The coradius formula therefore fails for two reasons in predicting the waiting time until convergence. First of all, the single mutation to reach $D(\{n+2\})$ requires a waiting time that increases exponentially with the size of the system. Second, overcoming the negative drift of the undisturbed dynamics between states 1 and n will require a waiting time that also increases exponentially in the size.

For the purpose of characterizing the long-run behavior of a dynamic system, stochastic stability analysis takes the wrong limit by fixing the size n of the system and letting $\epsilon\to 0$. While we typically think of the noise term ϵ as small, we also want it to be sufficiently bounded away from 0 such that the stochastic system does not get 'stuck' in intermediate limit sets in the medium run. At the same time we usually want our results to hold primarily for environments with many agents,

[59]The probability of finding the random walk in the δ-neighborhood can be calculated as in appendix C.

[60]Note, that otherwise $\left(\frac{1}{2}\right)^{n-1}\frac{1-\epsilon}{\epsilon}<\left(\frac{1}{2}\right)^{n-1}\left(\frac{3}{2}\right)^{n}\to 0$ as $n\to\infty$.

[61]The estimate for the waiting time is the product of the waiting time until a single mutation occurs (which is at least $\overline{\epsilon}_n^{-1}$) and the waiting time to reach the δ-neighborhood (which is of the order 2^n as the ratio of the probability for a downward-jump and the probability of an upward jump is 2 under the undisturbed dynamics).

due to the bounded rationality assumption buried in almost all evolutionary models. Agents behave myopically or use rules of thumb because their computational abilities are assumed to be limited. This simplification in the decision-making process is particularly compelling for models with local interaction, such as my residential neighborhood process, because the number of possible states increases exponentially in the size of the system.

Clustering describes the long-run and medium-run behavior of a stochastic process more adequately by taking the 'correct' limit $n \to \infty$. The perturbation ϵ is kept fixed such that clustering takes both the disturbed and the undisturbed dynamics of the process into account. In my simple example it can be easily checked that the process clusters around any δ-neighborhood of state 0. Moreover, the process reaches the neighborhood quickly as the waiting time satisfies $W(n, [0, \delta n]) \sim n$. Clustering therefore completely reverses the predictions of the standard analysis.

6.1 Example I: Formation of Black Ghettos on Streets

I next demonstrate that stochastic stability describes the evolution of the residential neighborhood process very poorly on streets, which vindicates my choice of clustering over the standard technique for the analysis of the model.

The only limit sets of the process on streets are the all-white and all-black configuration. It is straightforward to determine the radius and coradius of the black ghetto configuration η_b. The process will leave the basin of attraction of the ghetto only once a minimally stable white cluster of length $w = [2r(1 - \alpha_w)]^+$ has formed, i.e. at least w completely tolerant white house-seekers have settled on the street. Therefore, one can deduce that $R(\{\eta_b\}) = w$. Similarly, an all-white neighborhood can turn into a black ghetto with positive probability only in the presence of a minimally stable black cluster of length $b = [2r(1 - \alpha_b)]^+$ which tells us that $CR(\{\eta_b\}) = b$.

Because blacks are assumed to be more tolerant than whites, we know already that $b \leq w$. If blacks are sufficiently more tolerant and/or the radius of interaction r is sufficiently large the inequality becomes strict. In this case the black ghetto is stochastically stable according to theorem 5 and will be reached quickly in the medium run because the waiting time is $O(\epsilon^{-b})$ on a street of fixed size n.[62]

But these conclusions contradict the findings of section 3 where I showed that the process can cluster both around the all-black *and* the all-white configuration depending on the ethnic composition of the housing market. Intuitively, stochastic stability fails for exactly the same reasons as in the simple random walk example I considered previously. The radius/ coradius reasoning focuses on a 'sideshow' of

[62] The bound on the waiting time does not depend on the size of the system which is interpreted as 'fast' convergence.

the dynamics by looking at the emergence of minimally stable clusters of minority residents. Unless the share ϵ of tolerant agents is unrealistically small, such clusters will always arise quickly. For large and even moderately large streets, however, all the 'action' comes from the undisturbed dynamics which governs the evolution of the street *after* minimally stable clusters have formed. The ethnic composition of the housing market then determines whether white or black clusters can expand with positive drift. Clustering captures this effect, while stochastic stability does not.

6.2 Example II: Imitation and Coordination

The following simple model of imitation between communicating agents on a graph provides a further illustration for the weak predictive power of stochastic stability when applied to large-scale systems. It is also of interest in its own right because the result holds for a wide class of non-regular graphs.

I consider n agents who live on some connected graph of order q.[63] I refer to this class of graphs as 'proper' graphs. This can be a street, an inner-city area or some more complicated structure such as a simple street with an even number of agents where each agent has an additional randomly drawn third neighbor. Agents choose to hold exactly one of two possible 'opinions' which I denote 0 and 1. Time is continuous, and all agents revise their opinion each time their Poisson 'alarm clock' goes off at rate 1. With (small) probability ϵ they listen to an exogenous signal telling them to change their opinion. Otherwise, they sample one of their q neighbors and imitate her action.

Stochastic stability suggests that society should hold unanimous opinions most of the time. It takes one mutation to leave the unanimous configurations η_0 and η_1 where all agents hold either opinion 0 or opinion 1. On the other hand, a path leading to an unanimous configuration has cost 0. Therefore, the radius $R(\{\eta_0, \eta_1\})$ exceeds the coradius $CR(\{\eta_0, \eta_1\})$ and theorem 5 applies.

However, Monte Carlo simulations do not confirm this prediction, but suggest, to the contrary, that typical societies are perfectly 'confused'. For the numerical analysis I call a society 'unanimous' if the long-run share $\tilde{X}_n$ of agents with opinion 1 lies either in the interval $[0, 0.1]$ or the interval $[0.9, 1]$. Society is called 'confused' if the share $\tilde{X}_n$ lies in the interval $[0.45, 0.55]$, i.e. society is almost evenly divided into two camps. Table 5 compares the respective probability of society being unanimous or confused for streets $G_S^{r=2}(n)$ of varying size n and exogenous signals ϵ of different strength. Society is well described as unanimous only if both the size of the graph *and* the disturbance term ϵ are small. As the size of society increases agents hold very rarely the same opinions and society becomes more and more confused. 'Confusion' will be more pronounced if agents are more likely to listen

[63]A graph has order q if q edges meet at each node.

Table 5: Comparing the probability p_U that society is 'unanimous' with the probability p_C that society is 'confused' when agents imitate the opinions of their neighbors. Agents live on a street $G_S^{r=2}(n)$ of varying size n and listen to exogenous signals ϵ of different strength.

$\epsilon = 0.1$

	$n=10$	$n=50$	$n=100$	$n=200$	$n=1000$
p_U	0.326	0.002	0.000	0.000	0.000
p_C	0.104	0.275	0.423	0.552	0.899

$\epsilon = 0.01$

	$n=10$	$n=50$	$n=100$	$n=200$	$n=1000$
p_U	0.836	0.168	0.030	0.001	0.000
p_C	0.022	0.131	0.216	0.300	0.602

Estimated standard errors are 0.001 or less for all estimates. Society is called 'unanimous' if $\tilde{X}_n$ lies in the interval $[0, 0.1]$ or $[0.9, 1]$, and is called 'confused' if $\tilde{X}_n$ lies in the interval $[0.45, 0.55]$.

to the exogenous signal, but even if this event occurs very rarely ($\epsilon = 0.01$) society will be confused at least 60 percent of the time for $n \geq 1000$.

The failure of stochastic stability analysis can be again traced back to its inability to take into account the intermediate dynamics, and its overemphasis on the noisy dynamics close to the two unanimous configurations. This can be most clearly seen by considering the special case of a complete graph of size n where each agent has $n-1$ neighbors. For large n the process is well described by the deterministic approximation of the change in the share of agents $x(t)$ holding opinion 1:

$$\begin{aligned}
\frac{dx}{dt} &= \text{share agents with opinion } 0 \times \text{Prob. of switching from 0 to 1} \\
&- \text{share agents with opinion } 1 \times \text{Prob. of switching from 1 to 0} \\
&= (1-x)\left[\epsilon + (1-\epsilon)x\right] - x\left[\epsilon + (1-\epsilon)(1-x)\right] \\
&= \epsilon(1-2x) \qquad (10)
\end{aligned}$$

This differential equation has a unique stable steady state at $x^* = \frac{1}{2}$, which suggests that the imitation process clusters around x^* in the case of complete graphs. Note that the imitation effect cancels out because it is linear: the probability of changing one's mind is proportional to the number of neighbors with different opinion. The exogenous signal then pushes the process towards x^*.

The next theorem shows that this observation extends to *any* increasing sequence of proper graphs and that the process converges fast.[64]

Theorem 6 *Consider the imitation process on some sequence $G(n)$ of proper graphs. The share of agents holding opinion 1 clusters around any neighborhood of $x^* = \frac{1}{2}$. The process reaches some δ-neighborhood of x^* after a waiting time of $W(n, [x^* - \delta, x^* + \delta]) = O(1)$.*

Proof: see appendix I

If we believe that typical societies are large we should indeed observe them to be 'confused' most of the time. The numerical results in table 5 reassure us that societies do not have to be unduly large for theorem 6 to hold.

6.3 Example III: Revisiting Ellison's (1993) Model of Local Interaction and Coordination

Finally, I demonstrate how clustering can enhance our understanding of existing evolutionary models whose dynamics was characterized through stochastic stability analysis. Ellison (1993) examined how agents in large populations learn to play a 2×2 coordination game, shown in figure 15. Strategy A is assumed to be risk-dominant. Agents live on a street with radius of interaction r.[65] Time is discrete, and in each period t agents play with probability $1 - 2\epsilon$ the best response to the average play of their neighbors at period $t-1$. With probability 2ϵ they choose one of the two strategies A and B at random with 50-50 probability. This system has the two limit sets η_A and η_B with all agents playing strategy A and B, respectively.

Because of risk dominance, the best response of a player will be action A if at least some fraction $q^* < \frac{1}{2}$ of her neighbors play A. This implies that a cluster of at most $r + 1$ agents playing strategy A can expand contagiously under the undisturbed dynamics and take over the entire street. The limit set η_A has therefore a large basin of attraction, while the limit set η_B has a small one. The radius/ coradius reasoning then quickly establishes that all agents play A in the long run (see Ellison (1999)).

Stochastic stability predicts the same long-run behavior as clustering, as I will show shortly. Nevertheless, this prediction is not very robust because tiny changes in the dynamics of the model can make η_B the stochastically stable equilibrium. Consider, for example, the following modification: in each period an agent faces

[64]Note, that theorem 2 and theorem 3 only hold for an increasing sequence of streets and inner-cities, respectively.

[65]Kandori, Mailath and Rob (1993) look at the case of uniform interaction.

	A	B
A	a,a	c,d
B	d,c	b,b

Figure 15: Stage game with strategy A as the risk-dominant strategy: $(a - d) > (b - c)$

a different stage game with a very small probability $\sigma > 0$ where she receives an additional payoff g from playing B against a neighbor who also plays B. I assume that g is large enough, such that this agent would play B if at least one of her neighbors played B in the previous period. Intuitively, we would not expect this change to have a major effect on the behavior of the model. After all, a cluster of $r+1$ agents playing strategy A is still very likely to expand, even though the growth of the cluster is no longer contagious. But it still grows with a strong positive drift and the modified system very much resembles the residential neighborhood process on a street with the cluster of agents playing A corresponding to the minimally stable cluster of black residents and blacks dominating the housing market. The next theorem confirms this intuition.[66]

Theorem 7 *Consider Ellison's modified population coordination game on a street $G_S^r(n)$. There exists some critical value $0 < \tilde{\sigma} < 1$ such that the following holds for $\sigma < \tilde{\sigma}$.*

1. *Most agents play strategy A in the long run, i.e. the share of agents who play strategy A clusters on the interval $[x_A^*(\epsilon), 1]$ with $\lim_{\epsilon \to 0} x_A^*(\epsilon) = 1$.*

2. *For ϵ sufficiently small the waiting time until the share of agents playing strategy A exceeds $1 - \delta$ satisfies $W(n, [1 - \delta, 1]) = O(1)$.*

Proof: The proof is exactly analogous to the proofs of theorem 2 and lemma 1 in section 3.2 and is therefore omitted. Action A (B) corresponds to a resident being black (white) and the parameter σ plays the role of the balance in the housing market.

[66] The second part of the theorem contains theorem 3 in Ellison (1993) as a special case.

Note, that the theorem also holds for Ellison's original setup which corresponds to the special case $\sigma = 0$. Hence the small modification of the dynamics has no discontinuous effect on the evolution of the model. However, it has a dramatic effect on the basins of attraction of the two limit sets η_A and η_B. In particular, the size of the basin $D(\eta_A)$ is much smaller as the process can escape from configuration η_A as soon as two neighbors play strategy B. Therefore, η_B has a coradius $CR(\eta_B) = 2$ while the radius is still $R(\eta_B) = [q^*2r]+1$. If the radius of interaction is sufficiently large, the configuration η_B, rather than η_A, is stochastically stable according to theorem 5.

The sensitivity to innocuous changes in the dynamics of a model is a worrisome feature of stochastic stability. The problem arises, because the standard technique ignores the nature of the dynamics *after* the process has left the basins of attraction of the limit sets η_A and η_B. This part of the dynamics is not significantly influenced by small changes in σ, which explains why the predictions of clustering remain unaffected.

7 Conclusion

This paper outlines a new theory to understand the rise and the persistence of ghettos in US cities. I build a simple evolutionary model which is completely described by the geometry of the residential area, the tolerance level of both ethnic groups and the balance in the housing market. I analyze in what way these parameters interact to lead to rapid segregation and found that the balance in the housing market is the determining factor. Furthermore, I prove that black ghettos can be very persistent in large inner-city areas.

My model can be viewed as a version of Schelling's (1972) tipping model with a richer non-uniform geometry of interaction. Exploring the implications of a local interaction set-up has been the domain of game theorists rather than applied researchers.[67] This is regrettable for two reasons. First, most social networks are local in the sense that the vast majority of agents interact with and care about only a small subset of the population. Local networks are therefore a far more natural modeling environment than the uniform geometry. Second, many surprising and empirically significant effects arise from the local interaction setup. One of the first insights of this kind was Ellison's (1993) observation that local interaction can hugely speed up the convergence to the long-run equilibrium. In the context of my model this effect manifests itself in the different mechanisms that uphold a black ghetto on a street compared to a bounded neighborhood. In the uniform

[67] Important exceptions are the work by Glaeser, Sacerdote and Scheinkman (1996) on crime and social interaction and by Möbius (1999) on competition in the telephone industry around the turn of the century.

geometry a black ghetto is persistent because white residents feel isolated and refuse to enter the residential area. On streets, on the other hand, the ghetto is upheld because blacks dominate the housing market. "Avenue waves" such as observed in Chicago are only possible within the local interaction setting.

I hope that the new techniques developed in this paper will facilitate the analysis of models with local interaction. The standard radius/ coradius reasoning is, in some sense, too successful in simplifying the analysis of a dynamic model. It ignores a great deal of information about the intermediate dynamics and overemphasizes the dynamics around the limit sets. For large-scale systems this imbalance can lead to poor predictions of the medium- and long-run behavior of a process. In these cases clustering can prove to be a safer and more robust tool to understand the evolution of the system.

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A Transforming the Residential Process into a Discrete Time Markov Chain

It is often easier to work with the discrete time counterpart of a continuous-time Markov process on the state set Z. In particular, results from the stochastic stability literature can be applied directly.

The discrete time Markov process is constructed as follows. The discrete 'clock' is scaled so that time increases in increments of $\frac{1}{n}$. In each period exactly one of the n residents in the residential is randomly selected, moves out and is replaced by a newcomer from the housing market. The switching functions are assumed to be the same as for the continuous-time process. I introduce the convention that the configuration η_z is obtained from η by inverting the ethnicity of the resident at size z and leaving the other residents unchanged. The transition matrix P^ϵ then becomes:

$$P^\epsilon(\eta,\mu) = \begin{cases} \frac{1}{n} g_w^\epsilon(x(\eta,z)) & \text{if } \mu = \eta_z\ ,\ \eta(z) = 0 \\ \frac{1}{n} g_b^\epsilon(y(\eta,z)) & \text{if } \mu = \eta_z\ ,\ \eta(z) = 1 \\ 1 - \frac{1}{n}\sum_{i=1}^n (1-\eta(z_i))\, g_w^\epsilon(x(\eta,z_i)) - & \\ -\frac{1}{n}\sum_{i=1}^n \eta(z_i)\, g_b^\epsilon(y(\eta,z_i)) & \text{if } \mu = \eta \\ 0 & \text{otherwise} \end{cases}$$

Note, that each site will be chosen once per time unit just as in the continuous-time process. For this reason the long-run ergodic distribution and all waiting times derived for the discrete time model are the same as for the original continuous-time process.

The 'undisturbed' transition matrix P is obtained from P^ϵ by setting $\epsilon = 0$. The triple (Z, P, P^ϵ) describes a model of evolution with noise as specified by Ellison (1999) and his results for characterizing waiting times apply.

B Results on Random Walks with Drift

For the following theorem I assume that time is discrete and that time increases in increments of $\frac{1}{n}$ as in appendix A.

Lemma 5 *Consider a random walk on the integers between* 0 *and* $n > 0$*. The process moves up with probability* α *and down with probability* β *where* $\alpha + \beta \leq 1$ *and* $\alpha > \beta$*. Starting from* $0 < k < n$ *the process will reach* 0 *before it reaches* n *with probability*

$$p_k = \frac{\left(\frac{\beta}{\alpha}\right)^k - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^n}.$$

The waiting time of reaching n *- conditional on* n *being reached before* 0 *- is bounded above by* $\frac{1}{\alpha-\beta}+o\left(\frac{1}{n}\right)$.

Proof: The conditional probability p_k has to fulfill the following standard difference equation for $0<k<n$:

$$\begin{aligned} p_k &= \alpha p_{k+1}+\beta p_{k-1}+(1-\alpha-\beta)\,p_k \\ p_n &= 0 \\ p_0 &= 1 \end{aligned} \qquad (11)$$

For the second part of the lemma denote the conditional waiting time (measured in discrete time periods) starting from $0<k\leq n$ with w_k. The following equations have to be fulfilled (for the last one remember that the process is conditioned *not* to jump to 0):

$$\begin{aligned} w_k &= \alpha\,(w_{k+1}+1)+\beta\,(w_{k-1}+1)+(1-\alpha-\beta)\,(w_k+1) \\ w_n &= 0 \\ w_1 &= \frac{\alpha}{1-\beta}\,(w_2+1)+\frac{1-\alpha-\beta}{1-\beta}\,(w_1+1) \end{aligned} \qquad (12)$$

This system can be solved such that $w_1\approx\frac{n}{\alpha-\beta}+const$. For the result to follow note, that $w_k\leq w_1$ and that each discrete time period has duration $\frac{1}{n}$. QED

C Proof of Theorem 1

Because of the global geometry the space of configurations can be collapsed onto the reduced state space $Z'=\left\{0,\frac{1}{n},\frac{2}{n},..,1\right\}$ representing the possible share of blacks in the bounded neighborhood. The discrete counterpart of the residential neighborhood process is now described by the following Markov matrix P^ϵ (see appendix A for construction):

$$P^\epsilon(x,x')=\begin{cases}(1-x)\,g_w^\epsilon(x) & \text{if } x'=x+\frac{1}{n}\\ x g_b^\epsilon(1-x) & \text{if } x'=x-\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}$$

The process has an ergodic distribution μ_n on a bounded neighborhood of size n. For notational convenience I denote the probability of jumping from $\frac{m}{n}$ to $\frac{m-1}{n}$ with a_m and the probability of jumping from $\frac{m}{n}$ to $\frac{m+1}{n}$ with b_m. The ergodic distribution then has to fulfill the stationarity condition:

$$\begin{aligned}
(a_m + b_m)\,\mu_n\left(\frac{m}{n}\right) &= a_{m+1}\mu_n\left(\frac{m+1}{n}\right) \\
&+ b_{m-1}\mu_n\left(\frac{m-1}{n}\right) \quad \text{for } 0 < m < n \\
a_1\mu_n\left(\frac{1}{n}\right) &= b_0\mu_n(0) \\
b_{n-1}\mu_n\left(\frac{n-1}{n}\right) &= a_n\mu_n(1)
\end{aligned} \tag{13}$$

One can then show by induction:

$$\mu_n\left(\frac{m}{n}\right) = \frac{a_{m+1}}{b_m}\mu_n\left(\frac{m+1}{n}\right) \tag{14}$$

Recall, that there are at most three black shares where $a_m = b_m$:

$$\begin{aligned}
x_1 &= \tfrac{(1-\lambda)\epsilon}{(1-\lambda)\epsilon+\lambda} \\
x_2 &= 1-\lambda \qquad \text{if } 1-\alpha_w < \lambda < \alpha_b \\
x_3 &= \tfrac{1-\lambda}{1-\lambda+\lambda\epsilon}
\end{aligned}$$

It can be easily checked that the random walk exhibits a drift towards x_1 on $B_1 = [0, 1-\alpha_b)$, towards x_2 on $B_2 = (1-\alpha_b, \alpha_w)$ and towards x_3 over $B_3 = (\alpha_w, 1]$.

This observation is sufficient to show that the process will be found with probability approaching 1 inside any neighborhood of $\{x_1, x_2, x_3\}$. Consider any δ-neighborhood of x_1 for example (i.e. $I = (x_1 - \delta, x_1 + \delta)$). For $x < x_1 - \frac{\delta}{2}$ one can deduce that

$$\frac{a_{m+1}}{b_m} \leq C_{\epsilon,\delta} < 1,$$

while for $x_1 + \frac{\delta}{2} < x < 1 - \alpha_b$ the following holds:

$$\frac{b_m}{a_{m+1}} \leq C'_{\epsilon,\delta} < 1.$$

Let $\tilde{C}_{\epsilon,\delta} = \max\left(C_{\epsilon,\delta}, C'_{\epsilon,\delta}\right)$. This implies that $\mu_n(x) \leq \left[\tilde{C}_{\epsilon,\delta}\right]^{\frac{\delta n}{2}}$ for any $x \in B_1 - I$. Therefore the probability of finding the process inside $B_1 - I$ is at most

$$(1 - \alpha_b - 2\delta)\, n \left[\tilde{C}_{\epsilon,\delta}\right]^{\frac{\delta n}{2}},$$

which tends to 0 as $n \to \infty$. The same exercise can be repeated for x_2 and x_3 which establishes the claim.

Next, I show that the process has a vanishing probability weight around x_1. Using formula 14 repeatedly one can derive:[68]

$$\begin{aligned} \mu_n(x_1) &= \mu_n(x_3) \prod_{m=x_1 n}^{(1-\alpha_b)n} \frac{\lambda \frac{m}{n}}{(1-\lambda)\left(1-\frac{m}{n}\right)\epsilon} \times \\ &\times \prod_{m=(1-\alpha_b)n}^{\alpha_w n} \frac{\lambda \frac{m}{n}}{(1-\lambda)\left(1-\frac{m}{n}\right)} \prod_{m=\alpha_w n}^{x_3 n} \frac{\lambda \frac{m}{n}\epsilon}{(1-\lambda)\left(1-\frac{m}{n}\right)} \end{aligned} \tag{15}$$

We know that $1-x_1 < x_3$ as $\lambda < \frac{1}{2}$ which allows us to simplify the expression:

$$\mu_n(x_1) = \mu_n(x_3)\left(\frac{\lambda}{1-\lambda}\right)^{(1-2x_1)n} \epsilon^{(\alpha_b-\alpha_w)n} \prod_{m=(1-x_1)n}^{x_3 n} \frac{\epsilon\lambda \frac{m}{n}}{(1-\lambda)\left(1-\frac{m}{n}\right)} \tag{16}$$

Note, that $\epsilon\lambda x_3 \leq (1-\lambda)(1-x_3)$. Therefore every term in the product on the left is less than 1 and one obtains the inequality:

$$\mu_n(x_1) \leq \mu_n(x_3)\left[\left(\frac{\lambda}{1-\lambda}\right)^{(1-2x_1)n} \epsilon^{\alpha_b-\alpha_w}\right]^n \tag{17}$$

As $\alpha_b > \alpha_w$ or $\lambda < \frac{1}{2}$ one finds again that

$$\mu_n(x_1) \leq [F_\epsilon]^n$$

for $F_\epsilon < 1$. More generally, one can repeat the exercise for any x' in some small δ-neighborhoods of x_1 because the inequality $1-x_1 < x_3$ is strict. One then obtains

$$\mu_n(x') \leq [F_{\epsilon,\delta}]^n$$

for $F_{\epsilon,\delta} < 1$. As before this implies that the probability weight inside the δ-neighborhood vanishes as $n \to \infty$.

It remains to be determined whether the process clusters around x_2 or around x_3. Using formula 14 again one obtains:

$$\mu_n(x_2) = \mu_n(x_3)\left(\frac{\lambda}{1-\lambda}\right)^{(x_3-x_2)n} \prod_{m=x_2 n}^{x_3 n} \frac{\frac{m}{n}}{1-\frac{m}{n}} \quad \epsilon^{(x_3-\alpha_w)n} \tag{18}$$

[68]All expressions hold up to an integer constraint. As n becomes large the finite number of misplaced terms in the product have a vanishing influence and are therefore omitted.

We now use the fact that[69]

$$\left[\prod_{m=x_2n}^{x_3n}\frac{\frac{m}{n}}{1-\frac{m}{n}}\right]^{\frac{1}{n}}=\frac{x_3^{x_3}\left(1-x_3\right)^{1-x_3}}{x_2^{x_2}\left(1-x_2\right)^{1-x_2}}+o\left(\frac{1}{n}\right).$$

This implies, that

$$\mu_n\left(x_2\right)=\mu_n\left(x_3\right)\left[\tilde{F}_\epsilon\right]^n \tag{19}$$

where

$$\begin{aligned}\tilde{F}_\epsilon &= \left(\frac{\lambda}{1-\lambda}\right)^{x_3-x_2}\frac{x_3^{x_3}\left(1-x_3\right)^{1-x_3}}{x_2^{x_2}\left(1-x_2\right)^{1-x_2}}\quad\epsilon^{x_3-\alpha_w}+o\left(\frac{1}{n}\right)\\ &= \frac{\epsilon^{1-\alpha_w}}{1-\lambda+\lambda\epsilon}+o\left(\frac{1}{n}\right).\end{aligned}$$

As before one can show that the process vanishes on some δ-neighborhood of x_2 (x_3) if $\tilde{F}_\epsilon<1$ ($\tilde{F}_\epsilon>1$). The process clusters then over x_3 (x_2).

Finally, it is easy to show that the medium-run behavior of the process is determined by the initial conditions alone. Consider for example the case where the initial share of black residents satisfies $x_1+\delta<x_0<1-\alpha_b$. Using lemma 5 one can deduce that the probability of reaching $1-\alpha_b$ before reaching $x_1+\delta$ goes to zero exponentially as $n\to\infty$ and the conditional waiting time for reaching the δ-neighborhood is bounded above by some finite W_δ. QED

D Proof of Lemma 2

In both marginal processes a switch at some cell z is associated with an 'event'. For the original process η_t that event is simply the switch of color at that particular cell. For the simplified process σ_t on the other hand only a white to black switch always involves just one cell. A black to white switch might cause a number of cells to the right or left to switch too, and in this case I call the joint switching of all those cells the event corresponding to the switch at z.

Think of two identical streets $G_S\left(n\right)$ such that σ_t moves on street A and η_t on street B. At any point in time and for any cell z the cells on both streets switch

[69]Note, that

$$\left[\prod_{m=x_2n}^{x_3n}\frac{\frac{m}{n}}{1-\frac{m}{n}}\right]^{\frac{1}{n}}=\exp\left(\int_{x_2}^{x_3}\ln\frac{t}{1-t}dt\right)+o\left(\frac{1}{n}\right).$$

as follows. If $\sigma_t(z) \neq \eta_t(z)$ the cells will switch independently and trigger off the associated event at the rates specified by the switching rule. If $\sigma_t(z) = \eta_t(z)$ both cells will flip together and trigger off the associated events with as large a rate as possible consistent with the requirement that each process flips at the correct rate. In other words if z switches on street A with rate c_1 and on street B with rate c_2 (WLOG assume $c_1 < c_2$) then they flip together with rate c_1 and on street B the cell will additionally flip at an independent rate of $c_2 - c_1$. This procedure defines a coupled process (σ_t, η_t) with the correct marginal processes σ_t and η_t.[70]

It remains to be shown that $\sigma_t \leq \eta_t$ with probability 1 at any point in time t. At random times t_i $(i = 0, 1, 2, ..)$ cells on street A or street B switch $(t_0 = 0)$. There are two types of switches - joint flips which trigger off the associated events simultaneously on both streets and independent switches which involve either only street A or street B. I show by induction on i that $\sigma_{t_i} \leq \eta_{t_i}$ for all $i \geq 0$ which implies of course that $\sigma_t \leq \eta_t$ for all $t \geq 0$.

$i = 0$ The claim is true by assumption because $\sigma_0 = \eta_0$.

$i \rightarrow i+1$ Assume the claim holds for i such that $\sigma_{t_i} \leq \eta_{t_i}$. First, assume the switch at time t_{i+1} at cell z is an independent switch on either street A or street B. If $\sigma_{t_i}(z) \neq \eta_{t_i}(z)$ we must have $\sigma_{t_i}(z) = 0$ and $\eta_{t_i}(z) = 1$. The associated event then involves in any case just the cell z and therefore $\sigma_{t_{i+1}} \leq \eta_{t_{i+1}}$. If $\sigma_{t_i}(z) = \eta_{t_i}(z) = 1$ the independent switch must have occurred on street A because the flip rate from black to white is increasing in the share of white neighbors of cell z and $y(\sigma_{t_i}, z) \geq y(\eta_{t_i}, z)$. The associated event flips one or more cells on street A from black to white such that $\sigma_{t_{i+1}} \leq \eta_{t_{i+1}}$. If $\sigma_{t_i}(z) = \eta_{t_i}(z) = 0$ the independent switch must have occurred on street B for analogous reasons. Cell z is now occupied by a black resident such that again $\sigma_{t_{i+1}} \leq \eta_{t_{i+1}}$.
Finally assume that the switch at time t_{i+1} at cell z is a simultaneous switch on both streets. If $\sigma_{t_i}(z) = \eta_{t_i}(z) = 1$ then on street A possibly some additionally cells switch from black to white so that certainly $\sigma_{t_{i+1}} \leq \eta_{t_{i+1}}$. If $\sigma_{t_i}(z) = \eta_{t_i}(z) = 0$ only the cell z on both streets changes from white to black so that again $\sigma_{t_{i+1}} \leq \eta_{t_{i+1}}$. This proves the inductive hypothesis. QED

E Proof of Lemma 3

I will prove the lemma in detail for simple streets $(r = 1)$ where the minimally stable cluster has size 1. At the end I sketch how the proof generalizes to streets with radius of interaction greater than 1.

I use coupling once more in order to further simplify the process σ_t which is now restricted to a single segment. I start by replacing the space of configurations

[70]This coupling is a more elaborate variant of the basic Vasershtein coupling for spin systems (Liggett 1985, chapter 3).

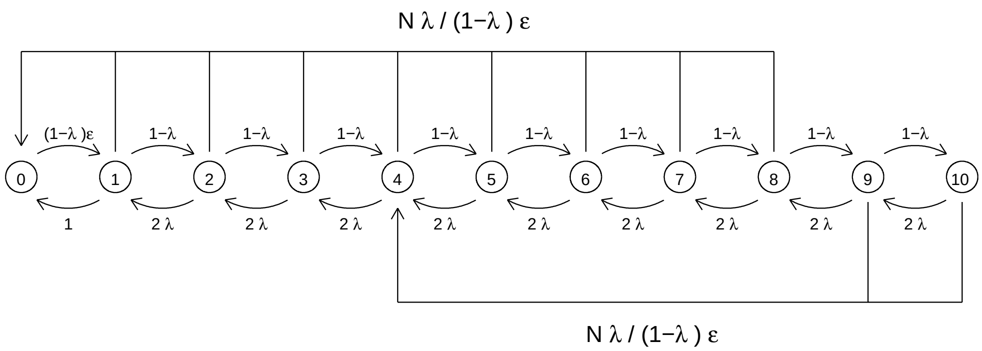

Figure 16: Dynamics of the random walk ξ_t^δ for $N = 10$ and $\delta = 0.1$

Z over which σ_t evolves. Because each segment contains at most one black cluster by construction, a configuration is completely described by the share of blacks in the set $Z' = \left\{0, \frac{1}{N}, .., 1\right\}$ and the position of the first clockwise black resident in the set $\tilde{Z} = \{1, 2, .., N\}$. We can then say that σ_t evolves on $Z' \times \tilde{Z}$.

For a fixed $\delta > 0$ I construct a continuous-time random walk ξ_t^δ on Z' with the following transition rate $c_\xi(x_1, x_2)$ between the states of Z':

$$c_\xi(x_1, x_2) = \begin{cases} (1-\lambda)\epsilon & \text{if } x_1 = 0 \text{ and } x_2 = \frac{1}{N} \\ 1-\lambda & \text{if } x_1 > 0 \text{ and } x_2 = x_1 + \frac{1}{N} \\ 2\lambda & \text{if } x_1 > \frac{1}{N} \text{ and } x_2 = x_1 - \frac{1}{N} \\ 1 & \text{if } x_1 = \frac{1}{N} \text{ and } x_2 = 0 \\ N\frac{\lambda}{1-\lambda}\epsilon & \text{if } x_1 \geq 1-\delta \text{ and } x_2 = \frac{1-\delta}{2} \\ N\frac{\lambda}{1-\lambda}\epsilon & \text{if } x_1 < 1-\delta \text{ and } x_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Figure 16 illustrates the dynamics of this random walk. It resembles a simple random walk with a drift determined by λ with the added capability of making 'large' jumps. With probability $N\frac{\lambda}{1-\lambda}\epsilon$ the process jumps to an intermediate state $\frac{1-\delta}{2}$ if the share of black residents is larger than $1-\delta$ and to 0 otherwise.

The random walk ξ_t^δ and the process σ_t are now coupled in the following way. Both evolve from the initial states 0 and $(0, 1)$ respectively, i.e. from an 'all-white' configuration. Any transition which increases (decreases) the share of blacks ξ_t^δ and $X(\sigma_t)$ is called an 'upward' jump ('downward' jump). I further distinguish between (small) downward jumps due to the undisturbed dynamics ('normal jumps') and those (potentially large ones) which are caused by the disturbance ('ϵ-jumps'). For the coupled process $\left(\xi_t^\delta, \sigma_t\right)$ the transitions are linked by the following rules:

1. Both processes jump upwards independently if $\xi_t^\delta \neq X(\sigma_t)$ and otherwise

jump simultaneously with as large a possible rate consistent with the requirement that both jump at the correct rate.

2. For normal downward jumps adopt the same convention as for upward jumps. For downward ϵ-jumps let both processes jump simultaneously with as large a possible rate consistent with the requirement that both jump at the correct rate.

The definition of the random walk ξ_t^δ ensures that the share of blacks is always less likely to increase and more likely to contract compared to the process σ_t. Using the same technique as in appendix D one can then show that coupling preserves the inequality $\xi_t^\delta \leq X(\sigma_t)$ at any point in time with probability 1.

Therefore, it is sufficient to show that there exists $\overline{\epsilon}$ such that for all $\epsilon < \overline{\epsilon}$ the expected long-run share of blacks $E_b(\epsilon)$ of the random walk ξ_t^δ fulfills $E_b(\epsilon) > 1-2\delta$ for some N. The result can only be true if the random walk has a positive drift i.e. $\lambda < \frac{1}{3} \leq \hat{\lambda}$. Denote the expected waiting time to jump out of the interval $(1-\delta, 1]$ starting from $x = 1$ with W_{out}. In this case the process can be found either at $x = 1-\delta$ or at $x = \frac{1-\delta}{2}$. Hence the expected waiting time to reach again the state $x = 1$ is at most the expected waiting time to get from $\frac{1-\delta}{2}$ to $x = 1$ which I denote with W_{in}. The expected long-run share of blacks can then be bounded below as follows:[71]

$$E_b(\epsilon) \geq (1-\delta) \frac{W_{out}}{W_{in} + W_{out}} \tag{20}$$

We now just have to find a lower bound for W_{out} and an upper bound for W_{in}.

I set $\epsilon = N^{-4}$. For large N the waiting time to jump out of the interval $(1-\delta, 1]$ through single downward jumps grows exponentially with N due to the positive drift of the random walk while the waiting time to leave the interval through an ϵ-jump is just $\frac{1-\lambda}{N\lambda\epsilon}$ which is only of the order $O(N^3)$. Therefore only the latter event matters and W_{out} can be bounded below as follows:

$$W_{out} \geq \frac{1}{2} \frac{1-\lambda}{\lambda} N^3 \tag{21}$$

Using the same techniques as in appendix B the waiting time $\hat{W}$ for reaching $x = 1$ starting from $\frac{1-\delta}{2}$ *conditional on no ϵ-jumps occurring* is bounded above by $\frac{N}{1-3\lambda} = AN$.[72] The probability that an ϵ-jump occurs is at most

$$AN \times N \frac{\lambda}{1-\lambda} \epsilon,$$

[71] The process spends at least a share $\frac{W_{out}}{W_{in}+W_{out}}$ of the time in the interval $(1-\delta, 1]$.

[72] More precisely, it is $\frac{N(1-\delta)}{2(1-3\lambda)} + O\left(N^{-1}\right)$.

which is of the order $O\left(N^{-2}\right)$. In the worst case the process ends up at $x=0$ after such an ϵ-jump. The waiting time to reach $x=1$ conditional on no further ϵ-jump occurring can the be calculated as[73]

$$\frac{1}{(1-\lambda)(1-3\lambda)\epsilon}+\frac{N}{1-3\lambda}+O\left(N^{-1}\right).$$

Because ϵ-jumps can only occur for $x>0$ the probability for such an event is as before of the order $O\left(N^{-2}\right)$. Therefore the unconditional waiting time $\tilde{W}$ to reach $x=1$ starting from $x=0$ is bounded above by

$$\frac{1}{1-O\left(N^{-2}\right)}\times \quad \text{conditional waiting time.}$$

The various estimates allow me to bound the waiting time W_{in} from above as follows:

$$\begin{aligned} W_{in} &\leq \hat{W}+AN^2\frac{\lambda}{1-\lambda}\epsilon\tilde{W} \\ &\leq AN+AN^2\frac{\lambda}{1-\lambda}\epsilon\times 2\left[\frac{A}{(1-\lambda)\epsilon}+AN\right] \\ &\leq AN+2A^2N^2\frac{\lambda}{(1-\lambda)^2}+2A^2\frac{\lambda}{1-\lambda}\frac{1}{N} \end{aligned} \tag{22}$$

Plugging the bounds for W_{out} and W_{in} into expression 20 delivers:

$$E_b(\epsilon)\geq(1-\delta)\left(1-O\left(N^{-1}\right)\right) \tag{23}$$

for $\epsilon=N^{-4}$. But this proves the lemma: simply choose $\overline{N}$ large enough such that $E_b(\epsilon)>1-2\delta$ for all $N>\overline{N}$ and take $\overline{\epsilon}=\overline{N}^{-4}$.

Finally I briefly discuss how to generalize the proof to radii of interaction $r>1$. There are two complications. First, the single black cluster on the segment grows by increments of 1 but can shrink by up to r in a single transition under the undisturbed dynamics of the process σ_t. The share of whites λ in the housing market therefore has to be low enough such that the associated random walk ξ_t^δ has a positive drift. Second, the size of the minimally stable cluster is now generally some $b>1$ and the dynamics requires b 'mutations' in order to jump out of the basin of attraction of $x=0$. The approximation 22 for W_{in} will not work any longer as ϵ does not cancel. This problem can be overcome by choosing more

[73] The first term captures the waiting time to jump out of $x=0$ corrected for the fact that the process visits $x=0$ on average $\frac{1}{1-3\lambda}$ times before reaching $x=1$. The second term corresponds to the simple waiting time for reaching $x=1$ if the process would start at $x=\frac{1}{N}$.

intermediate states for the random walk ξ_t^δ to which the process can move after a downward ϵ-jump. Simply define:

$$\begin{aligned} x_1^* &= \frac{1-\delta}{2} \\ x_2^* &= \frac{x_1^*-\delta}{2} \\ &\ldots \\ x_b^* &= \frac{x_{b-1}^*-\delta}{2} \end{aligned}$$

I then postulate that after an ϵ-event the random walk jumps to x_1^* if $\xi_t^\delta > 1-\delta$ and to x_i^* if $x_{i-2}^* - \delta \geq \xi_t^\delta > x_{i-1}^* - \delta$ for $1 < i \leq b+1$ (I set $x_0^* = 1$ and $x_{b+1}^* = 0$). The process will therefore reach $x = 0$ from x_1^* before it reaches $x = 1$ only with a probability of order $O\left(\epsilon^b\right)$ such that my approximation for W_{in} holds again. QED

F Proof of Lemma 1

For the proof I exploit the results from the proof of theorem 2 by taking $\epsilon = N^{-4}$ again. Through judicious coupling I constructed a random walk ξ_t^δ such that $\xi_t^\delta \leq X\left(\eta_t\right)$ with probability 1. Therefore it is sufficient to prove the hypothesis of the lemma for k independent random walks ξ_t^δ and let $k \to \infty$.

The expected waiting time W until the random walk ξ_t^δ reaches $x = 1$ starting from a share of blacks $x = 0$ can be bounded using the results from the previous section:

$$W \leq B\epsilon^{-b} + AN$$

The random walk can be subsequently found outside the $\frac{\delta}{2}$-neighborhood of $x = 1$ with probability $O\left(N^{-1}\right)$ (see appendix E). This implies that after a time $\frac{2W}{\delta}$ the process has reached $x = 1$ once with probability $1 - \frac{\delta}{2}$ and can therefore be found inside the $\frac{\delta}{2}$ neighborhood with probability $1 - \frac{\delta}{2} - O\left(N^{-1}\right)$.[74]

For large k the normal approximation of the binomial distribution can be used to show that the share of blacks on the street is at least $1 - \frac{\delta}{2} - \frac{2}{\sqrt{k}} - O\left(\frac{1}{N}\right)$ with probability 0.95 after time $\frac{2W}{\delta}$.[75] So we simply choose $k > \frac{16}{\delta^2}$ and the share of blacks has to be at least $1 - \delta$ after time $\frac{2W}{\delta}$ with probability 0.95. Therefore the expected time to reach the δ neighborhood of $x = 1$ is at most $\frac{1}{1-0.95}\frac{2W}{\delta}$. QED

[74] A random variable X with expectation $E\left(X\right)$ has to fulfill $P\left(X \leq \frac{E(X)}{\delta}\right) \geq 1 - \delta$.

[75] For a random variable X the following relation holds: $P\left(\left|X - E\left(X\right)\right| \leq 2\sigma\right) \geq 0.95$, where σ denotes the standard deviation.

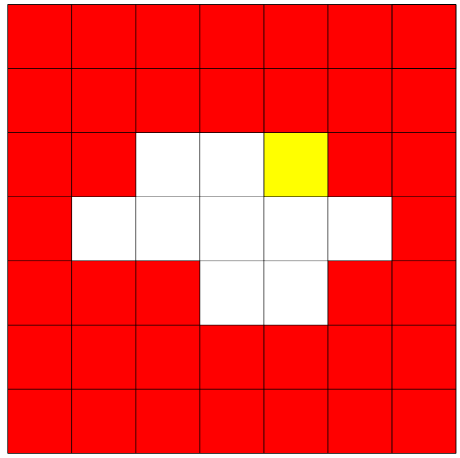

Figure 17: White "encircled" cluster in an inner-city with radius $r = 1$: white resident at upper right-hand corner (shaded) has two white and two black neighbors

G Proof of Lemma 4

Consider an encircled cluster C of white residents such as shown in figure 17 which forms the initial configuration of the residential neighborhood process, i.e. $\eta_0 = \chi_C$ where $\chi_C(z) = 0$ iff $z \in C$. By definition this cluster can be covered by a suitable rectangle S and $\eta_0 = \chi_C \geq \chi_S$.

First, I prove that $\eta_t \geq \chi_S$ for all $t > 0$ such that the white cluster can never 'break out' of S. This is equivalent to showing that after a sequence of individual residents switching at random times $t_0 = 0, t_1, t_2, .., t_i, ..$ the inequality $\eta_{t_i} \geq \chi_S$ holds. This claim is proved by induction. For $i = 0$ it is true by assumption. Assume it holds for $i - 1$. Then the switch from $\eta_{t_{i-1}}$ to η_{t_i} involves either a resident $z \in S$ or $z \notin S$. In the former case we are fine. The latter case is not possible, as for all $z \notin S$ and the configuration χ_S the share of black neighbors $x(\chi_S, z)$ of z is at least $\frac{1}{2} + \frac{r}{m}$. By the inductive hypothesis $\eta_{t_{i-1}} \geq \chi_S$ and hence $x\left(\eta_{t_{i-1}}, z\right) \geq x(\chi_S, z)$. Due to assumption 1 and the absence of tolerant agents no white house-seekers will ever be interested in moving to z.

Second, I prove that all configurations $\eta \geq \chi_S$ except the black ghetto configuration η_b are transient states of the Markov process. This establishes the result because the black ghetto is absorbing. It is sufficient to show that there exists a positive transition probability from any configuration $\eta \geq \chi_S$ to η_b. This will be guaranteed if for all configurations $\eta \geq \chi_S$ of cluster mass m there exists a positive transition probability to a configuration η' of cluster mass $m-1$ (i.e. a single white resident is replaced by a black resident) - in this case there exists with positive probability a sequence of m switches from η to η_b.

Consider a configuration $\eta \geq \chi_S$ of cluster mass $m > 0$. Choose any natural Cartesian coordinate system on the geometry G_C^r. Then find the upper right-hand corner z^* of the cluster (see figure 17). The share of white neighbors is at most $\frac{1}{2}$. Therefore, black house seekers will not feel isolated and show interest in the flat and the ethnicity of the resident at flat z^* will switch with positive probability of at least $1 - \lambda$. QED

H Proof of Theorem 4

It is sufficient to count the number of 'mutations' (i.e. completely tolerant house-seekers) that are necessary to leave the basin of attraction of the black ghetto configuration η_b. Ellison (1999) defines that number to be the radius $R(\eta_b)$ and the result follows directly from his theorem 2 (see appendix A for transforming the residential neighborhood process into an equivalent discrete time process).

On streets the result is trivial because a cluster of size $[2r(1-\alpha_w)]^+$ is minimally stable. In inner-cities it suffices to show that $\left[\frac{\sqrt{n}}{r+1}\right] - 1$ mutations do not allow the process to leave the basin of attraction of the black ghetto. The inner-city can be divided up into $k = \left[\frac{\sqrt{n}}{r+1}\right]$ full horizontal stripes H_i ($i = 1..k$) and equally many vertical stripes V_j ($j = 1..k$) of width $r+1$. Assume that $\left[\frac{\sqrt{n}}{r+1}\right] - 1$ tolerant agents moved into the inner-city up to time t. Then there must be at least one horizontal stripe H_{i^*} where no mutations has occurred. Due to assumption 1 all residents on that stripe have to be black (no white cluster could have invaded that stripe from outside). For the same reason there must be at least one vertical stripe V_{j^*} where all residents are still black at time t. But this implies that all existing white residents at time t can be covered by a rectangle of length and width at most $\sqrt{n} - r - 1$. Therefore, the set of whites is "encircled" and the configuration η_t is in the basin of attraction of the black ghetto by lemma 4. Hence at least $\left[\frac{\sqrt{n}}{r+1}\right]$ mutations are necessary to jump out of $D(\eta_b)$. QED

I Proof of Theorem 6

I will analyze the discrete time counterpart of the imitation process with transition matrix P^ϵ which can be derived as in appendix A. I index all the $\binom{n}{m}$ configurations $\eta \in Z$ where exactly m agents have opinion 1 by

$$\left\{\eta_{m,1}, \eta_{m,2} \ldots \eta_{m,\binom{n}{m}}\right\}.$$

I also introduce a reduced state space $Z' = \left\{0, \frac{1}{n}, \frac{2}{n}.., 1\right\}$ representing the possible shares of agents having opinion 1. The ergodic distribution μ_n^I of the imitation process on the graph $G(n)$ induces a corresponding distribution μ_n on the reduced state space defined by

$$\mu_n\left(\frac{m}{n}\right) = \sum_{i=1}^{\binom{n}{m}} \mu_n^I(\eta_{m,i}).$$

Next, I derive a transition matrix P_n describing a Markov chain on Z' which generates the ergodic distribution μ_n. For any configuration $\eta \in Z$ I index all $v(\eta)$ configurations which generate η when a single agent changes her opinion from 0 to 1 by

$$\left\{\eta^{-,1}, \eta^{-,2}...\eta^{-,v(\eta)}\right\}.$$

Similarly, I index all $w(\eta)$ configurations which generate η when a single agent changes her opinion from 1 to 0 by

$$\left\{\eta^{+,1}, \eta^{+,2}...\eta^{+,v(\eta)}\right\}.$$

For the ergodic distribution μ_n^I the probability 'outflow' from some configuration $\eta_{m,i} \in Z$ has to equal the probability 'inflow':[76]

$$\sum_{\mu \neq \eta_{m,i}} P^\epsilon(\eta_{m,i}, \mu)\, \mu_n^I(\eta_{m,i}) = \sum_{j=1}^{w(\eta_{m,i})} P^\epsilon\left(\eta_{m,i}^{+,j}, \eta_{m,i}\right) \mu_n^I\left(\eta_{m,i}^{+,j}\right) + \sum_{j=1}^{v(\eta_{m,i})} P^\epsilon\left(\eta_{m,i}^{-,j}, \eta_{m,i}\right) \mu_n^I\left(\eta_{m,i}^{-,j}\right)$$

After summing over the $\eta_{m,i}$ configurations one obtains:

$$\left(\frac{\sum_i \mu_n^I(\eta_{m,i}) \sum_j P^\epsilon\left(\eta_{m,i}, \eta_{m,i}^{-,j}\right)}{\mu_n\left(\frac{m}{n}\right)} + \frac{\sum_i \mu_n^I(\eta_{m,i}) \sum_j P^\epsilon\left(\eta_{m,i}, \eta_{m,i}^{+,j}\right)}{\mu_n\left(\frac{m}{n}\right)}\right) \mu_n\left(\frac{m}{n}\right)$$
$$= \frac{\sum_i \mu_n^I(\eta_{m+1,i}) \sum_j P^\epsilon\left(\eta_{m+1,i}, \eta_{m+1,i}^{-,j}\right)}{\mu_n\left(\frac{m+1}{n}\right)} \mu_n\left(\frac{m+1}{n}\right) + \frac{\sum_i \mu_n^I(\eta_{m-1,i}) \sum_j P^\epsilon\left(\eta_{m-1,i}, \eta_{m-1,i}^{+,j}\right)}{\mu_n\left(\frac{m-1}{n}\right)} \mu_n\left(\frac{m-1}{n}\right) \tag{24}$$

[76]Note, that the imitation process can reach $\eta_{m,i}$ only due to a single switch by one agent.

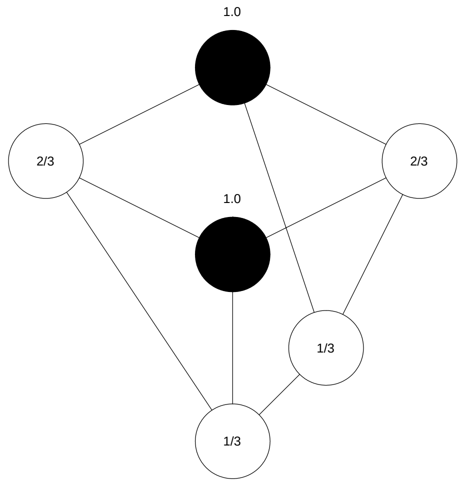

Figure 18: Proper graph of order $q = 3$ and a configuration with exposure $e(\eta) = 2$

I denote the probability of jumping from $\frac{m}{n}$ to $\frac{m-1}{n}$ with a_m and the probability of jumping from $\frac{m}{n}$ to $\frac{m+1}{n}$ with b_m such that equation 24 becomes:

$$(a_m + b_m)\,\mu_n\left(\frac{m}{n}\right) = a_{m+1}\mu_n\left(\frac{m+1}{n}\right) + b_{m-1}\mu_n\left(\frac{m-1}{n}\right)$$

The following transition matrix P_n gives rise to a Markov chain with ergodic distribution μ_n:

$$P_n(x, x') = \begin{cases} a_m & \text{if } x = \frac{m}{n} \text{ and } x' = \frac{m-1}{n} \\ b_m & \text{if } x = \frac{m}{n} \text{ and } x' = \frac{m+1}{n} \\ 0 & \text{otherwise} \end{cases}$$

The Markov chain described by P_n has the same structure as the process discussed in appendix C. I can therefore use formula 14:

$$\mu_n\left(\frac{m}{n}\right) = \frac{a_{m+1}}{b_m}\mu_n\left(\frac{m+1}{n}\right)$$

It is advantageous to transform the parameters in the following way. I define $\mu_n^*\left(\frac{m}{n}\right) = (a_m + b_m)\,\mu_n\left(\frac{m}{n}\right)$, $a_m^* = \frac{a_m}{a_m+b_m}$ and $b_m^* = \frac{b_m}{a_m+b_m}$. Formula 14 is preserved by these transformations:

$$\mu_n^*\left(\frac{m}{n}\right) = \frac{a_{m+1}^*}{b_m^*}\mu_n^*\left(\frac{m+1}{n}\right) \tag{25}$$

The ratio $\frac{a_{m+1}^*}{b_m^*}$ can be bounded by exploiting the linearity of imitation. For any configuration η I define weights w_z for each agent z on the graph such that whenever

(z, z') is an edge of the graph and agents z and z' use different actions then $\frac{1}{q}$ is added to the weights of both agents. These weights are exactly the switching probabilities of each agent under the undisturbed imitation process. From the construction and the properness of the graph it is immediately clear that the sum of weights of agents with opinion 0 equals the sum of weights of agents with opinion 1. This observation formalizes exactly the intuition that the imitation effect 'cancels out'. I call the sum of weights the *exposure* $e(\eta)$ of the configuration η. Clearly $0 \leq \frac{e(\eta)}{n} \leq \min(x, 1-x)$. Figure 18 demonstrates the algorithm on a small graph of order 3.

I can now conveniently express the probability that the process jumps from some given configuration η with m agents having opinion 1 to some configuration η' where $m+1$ agents have opinion 1:

$$(1-\epsilon)\frac{e(\eta)}{n} + \epsilon(1-x)$$

Similarly, the probability that the process jumps from η to η'' where $m-1$ agents have opinion 1 can be calculated as:

$$(1-\epsilon)\frac{e(\eta)}{n} + \epsilon x$$

I can then deduce

$$\begin{aligned} a_m &= (1-\epsilon)\, h(x) + \epsilon x \\ b_m &= (1-\epsilon)\, h(x) + \epsilon(1-x), \end{aligned} \tag{26}$$

where $h(x) = \frac{\sum_{i=1} \mu_n^I(\eta_{m,i}) e(\eta_{m,i})}{n\mu_n(m)}$. Note, that

$$0 \leq h(x) \leq x.$$

Hence, for $x < \frac{1}{2}$ the following inequalities must hold:

$$\begin{aligned} a_m^* &= \frac{(1-\epsilon)\, h(x) + \epsilon x}{2(1-\epsilon)\, h(x) + \epsilon} \leq \frac{1-\epsilon}{2} + \epsilon x \\ b_m^* &= \frac{(1-\epsilon)\, h(x) + \epsilon(1-x)}{2(1-\epsilon)\, h(x) + \epsilon} \geq 1-x \\ \frac{a_{m+1}^*}{b_m^*} &\leq \frac{\frac{1-\epsilon}{2} + \epsilon x}{1-x} + O\left(\frac{1}{n}\right) \end{aligned} \tag{27}$$

For $x < \frac{1}{2} - \frac{\delta}{2}$ the following uniform bound holds:

$$\frac{a_{m+1}^*}{b_m^*} \leq \frac{\frac{1-\epsilon}{2} + \epsilon\frac{1-\delta}{2}}{\frac{1+\delta}{2}} + O\left(\frac{1}{n}\right) \leq C_{\epsilon,\delta} < 1$$

Due to the symmetry of the ergodic distribution and repeated use of formula 25 one can conclude that $\mu_n^*(x) \leq [C_{\epsilon,\delta}]^{\frac{\delta n}{2}} \mu_n^*\left(\frac{1}{2} - \frac{\delta}{2}\right)$ for $\left|x - \frac{1}{2}\right| > \delta$. Recall, that $a_m + b_m > \epsilon$. Therefore the process will be found within the δ-neighborhood of $x^* = \frac{1}{2}$ with a probability of at least

$$1 - \frac{1-2\delta}{\epsilon} n \left[C_{\epsilon,\delta}\right]^{\frac{\delta n}{2}},$$

which tends to 1 as $n \to \infty$. Therefore, the process clusters around x^*.

The second part of the theorem is now easy. The waiting time until the δ-neighborhood is reached is at most as large as the corresponding waiting time of a process which satisfies

$$\frac{a_{m+1}^*}{b_m^*} = C_{\epsilon,\delta}$$

for $x < \frac{1}{2} - \frac{\delta}{2}$ and starts from $x = 0$. Because this random walk has a positive drift the waiting time is $O(n)$ (see appendix B). Note, that the discrete 'clock' increases at increments of $\frac{1}{n}$ and the result follows immediately. QED